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THE

PRINCIPLES AND PRACTICE

OF

NEAR PERSPECTIVE,

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DIVESTED OF ALL DIFFICULTY.

81173

By RICHARD ABBATT, F.R.A.S.

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TO THE RIGHT HONOURABLE

GEORGE WILLIAM FREDERICK HOWARD,

EARL OF CARLISLE,

WITH WHOSE NAME

THE FINE ARTS AND ELEGANT STUDIES OF LIFE

ARE INTIMATELY ASSOCIATED,

THESE PAGES ON

PERSPECTIVE DRAWING

ARE, WITH HIS LORDSHIP'S PERMISSION,

RESPECTFULLY INSCRIBED.

THE

ELEMENTS AND PRACTICE

OF

LINEAR PERSPECTIVE.

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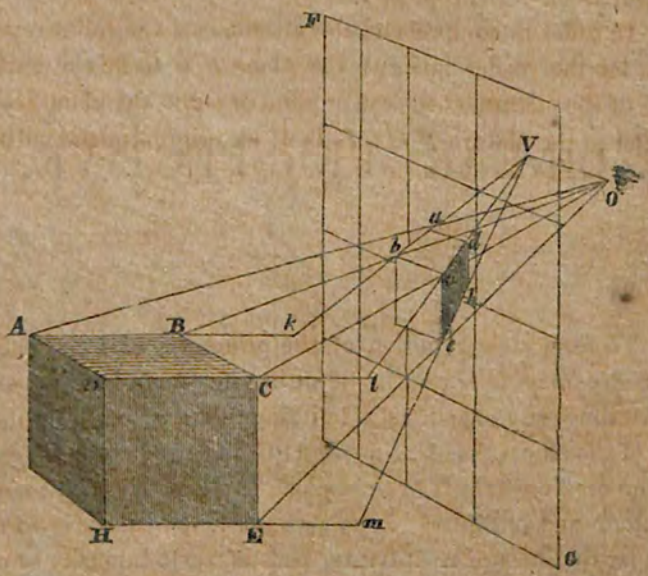
1. To form a clear notion of the principles of Perspective, let the reader imagine himself to be sitting in a room before a window, and viewing any object, as a building, a tree, &c. The different lines of the object may be supposed to be marked upon the glass by rays proceeding from it to the eye, and to form a *picture* complete in colour and in figure. If it were possible to transfer to canvas or paper the lines of such figure, and the same intensity of colour, brightness, and shade, as appear in the glass panes of the window, it would not be possible to detect the picture on the canvas from the natural appearance of the real object seen in its proper position. In order to imitate this as much as possible, it is necessary that the rays of light should come from the several parts of the picture to the eye of the spectator under the same circumstances of direction, strength of light, colour, and shade, as from the corresponding parts of the real object.

2. If we suppose the spectator, whose eye is at *O*, Fig. 1, to be viewing the cube *A B C D H E*, the picture *a b c d h e*, which is represented or depicted in idea upon the transparent plane *F G*,

is the true perspective representation of the cube, and may be correctly transferred to canvas or paper. The light from any point *a* of the picture ought to come to the spectator's eye at *O*, by the ray *a O* in the same direction, and with the same strength of light, &c., as it would do from the corresponding point *A* of the original cube by the ray *A O*. The eye at *O* in the annexed figure will, however, not see the same picture *a b c d h e* as that

Fig. 1.



which is seen by the reader, but the reverse figure, or that presented by viewing the back of the picture held up to the light. Drawing, therefore, which relates to the figures and lines of objects, and when done according to correct rules from mathematical principles, and not by an acquired habit of the hand and eye, is *Perspective*. These principles and rules, being few in number and very simple, we purpose fully to explain and apply to practice in the ensuing pages.

3. The reader may find it convenient to refer to the following properties of planes, as

Axioms in Perspective.

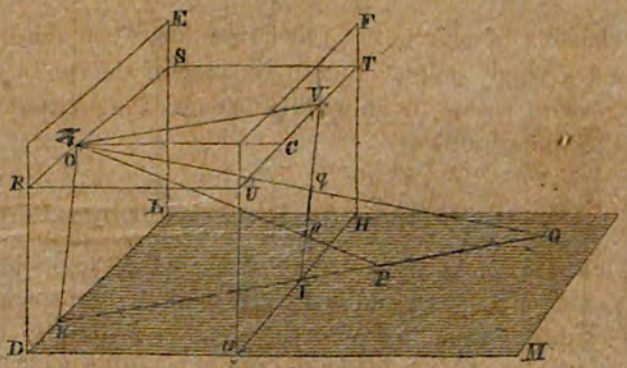
Every point in a straight line is in any plane that line is in.—*Euclid*, Book XI. Prop. 1.

The common section of any two planes is a straight line.—*Euclid*, Book XI. Prop. 3.

If three straight lines cut one another, or if two parallel straight lines are cut by a third, all the three will be in the same plane.—*Euclid*, Book XI. Prop. 2, 7.

4. In order to comprehend the meaning of the following definitions, let the reader imagine the plane $F\ G$ to be the surface or plane of the picture, O the eye or point of sight, the plane $D\ E$ to be parallel to the picture $F\ G$; $L\ D\ M$ an original plane cutting the

Fig. 2.



picture in the line $G\ H$, and its parallel plane in $D\ L$; $P\ Q$ an original line in the original plane $L\ D\ M$, intersecting the picture in I, and the plane $D\ E$ in K. And through the point of sight suppose a plane $R\ S\ T\ U$ to pass parallel to the original plane, $L\ D\ M$, cutting the picture in $U\ T$; also let the line $O\ V$ be drawn parallel to the original line $P\ Q$, and cut the picture in V; lastly, join $V\ I$, $O\ K$, $O\ P$, and $O\ Q$.

DEFINITIONS.

Projection, System of Rays, Cone of Rays, Optic Cone, Picture.

5. When lines, drawn according to a certain law, from the several parts of a figure, cut a plane, and by that cutting or intersection describe a figure on that plane, it is called a *projection* or representation of that figure. The lines producing that projection, taken together, are called a *system of rays*. When all those rays pass through the same point, they are called a *cone of rays*; and when that point is considered the eye of the spectator, the system of rays is called the *optic cone*.

The projection made by the optic cone is the *picture* of the object.

The shadows of objects are this projection, when the light is considered a single point. But in the case of the sun or moon, situated (as to sense) at an infinite distance, the projecting rays are parallel to each other.

Point of Sight, Centre, and Distance of the Picture.

6. The *point of sight** is that point where the eye ought to be placed to view the object, and is the vertex of the optic cone. If a line be drawn from the point of sight perpendicular to the picture, the point where it meets the picture is called the *centre of the picture* (*C*, Fig. 2). And the distance between the point of sight and the centre of the picture is called the *distance of the picture* (*O C*).

Intersection.

7. The line in which any original plane (produced, if necessary) cuts the picture is called the *intersection of that original plane*, as *G H*. And the point in which any original line meets the picture is called the *intersection of that original line*, as *I*.

* The centre of the picture is by artists sometimes called the point of sight.

Directing Plane, Line, Point; Director.

8. If through the point of sight there be imagined a plane to pass parallel to the picture, it is called the *directing plane*, as *E D*. The line in which any original plane cuts the directing plane is called the *directing line* of that original plane, as *D L*. And the point in which any original line cuts the directing line is called the *directing point* of that original line, as *K*. A line drawn through the directing point and the point of sight is called the *director* of that original line, as *O K*.

Parallel of Original Plane and of Original Line.

9. A plane passing through the point of sight parallel to any original plane is called the *parallel of that original plane*, as *R S T U*. And a line drawn through the point of sight parallel to any original line is called the *parallel of that original line*, as *O V*.

Vanishing Line, its Centre, and its Distance.

10. The line in which the parallel of any original plane cuts the picture is called the *vanishing* line of that plane, as *U V T*. And if from the point of sight there be drawn a line cutting the vanishing line at right angles, this point is termed the *centre of the vanishing line;* and the distance of this centre from the eye, or point of sight, is called the *distance of that vanishing line*.

If *O C* meet the vanishing line *U V T* at right angles in the point *C*, then *C* is the *centre* and *O C* the *distance* of that vanishing line.

Vanishing Point and its Distance.

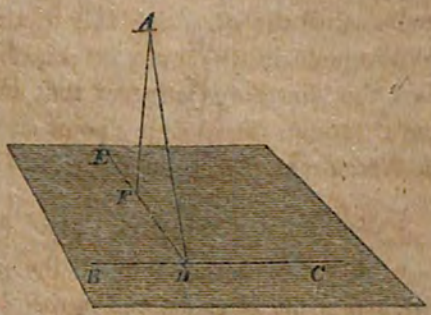
11. The point in which the parallel of any original line meets the picture is called the *vanishing point* of that original line, as *V*. And the distance of the vanishing point from the eye or point of sight is called the *distance of that vanishing point*, as *O V*.

Hence the distance of the vanishing point is equal to the distance between the directing point and the intersection.

PROPOSITION I.

12. To draw a straight line perpendicular to a given plane from a given point above it.

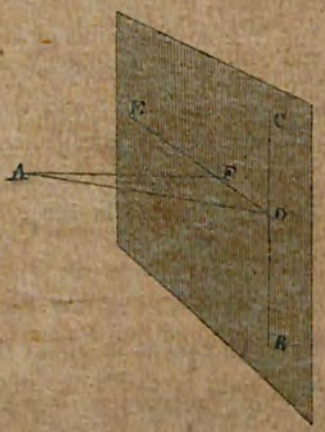
Fig. 3.



In the given plane draw any straight line, $B\ C$, and from the point A draw the straight line $A\ D$ perpendicular to $B\ C$. From the point D draw $D\ E$ in the plane at right angles to $B\ C$, and draw $A\ F$ from the point A perpendicular to $D\ E$; then $A\ F$ will be the perpendicular required. This is Euclid's construction, Book XI. Prop. 11.

PROPOSITION II.

Fig. 4.



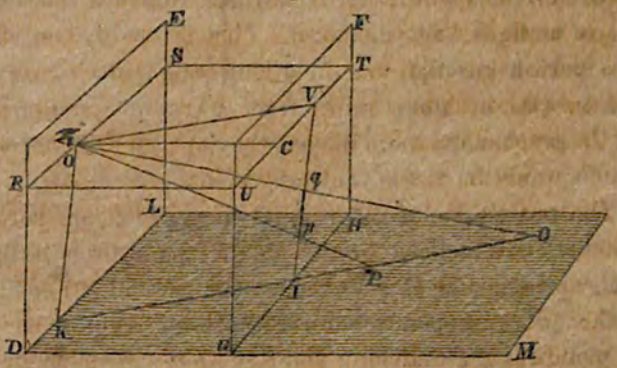
13. A line drawn from the centre of the picture to the centre of a vanishing line is perpendicular to that vanishing line.

Let A be the point of sight, F the centre of the picture, $B\ C$ the vanishing line of a plane, and D the centre of that vanishing line. Join $D\ F$; then, by the preceding proposition, $D\ F$ is perpendicular to $B\ C$.

PROPOSITION III.

14. The projection of a straight line, which is not parallel to the picture, passes through both its intersection and vanishing points.

Fig. 5.



Let $G\ H\ F$ represent the plane of the picture, O the point of sight, $p\ q$ the projection of the original line $P\ Q$ ($O\ P$, $O\ Q$ being the rays which, by cutting the picture, produce the projections of the points P, Q). Hence $p\ q$ is the intersection of the triangle $O\ P\ Q$ with the plane of the picture. But the line $Q\ P\ I$, and consequently the point I, is in that plane; hence $p\ q$ produced will pass through the point I.

Also $O\ V$ being parallel to $Q\ P$ (11) is likewise in the plane of the triangle $O\ P\ Q$; therefore $p\ q$ continued must pass through the point V.

In fact, all the points Q, P, I, K, O, V are in the same plane; and this plane, by its intersection with the picture, will cut the latter in the straight line $I\ p\ q\ V$, in which $p\ q$ is the projection of the original line $P\ Q$, I is the intersection, and V the vanishing point.

This proposition being the foundation of the practice of Perspective, the reader ought to make himself familiar with it. In Fig. 1 (page 2) he may see that the projections $a\ b$, $c\ d$, $h\ e$, of the three original lines $A\ B$, $D\ C$, $H\ E$ pass through the intersections k, l, m, and through the common vanishing point V.

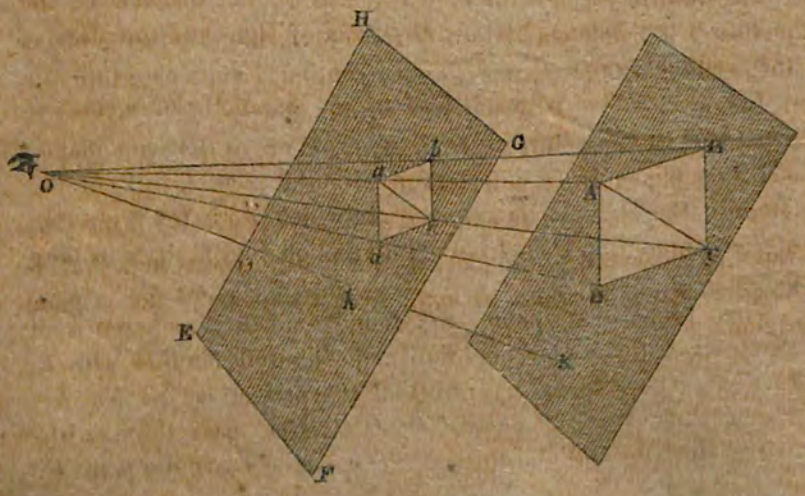
15. When the original line itself passes through the vanishing point, the whole projection of it will be in that point, and the line is said to vanish, which is one reason for using that term. Another reason is, that the further any object is off, the smaller is its projection, and the nearer it is to the vanishing point. When it comes into this point its magnitude vanishes, because the original is at an infinite distance. This is readily conceived by seeing a person go from you in a long walk; or viewing a train in motion, two or three miles from a railway station: these appear the smaller the more distant they are, and the nearer their projections would be to the vanishing point.

16. The projections of all original lines which are parallel to one another, but inclined to the picture, pass through the same vanishing point. For they have but one parallel (9) common to them all, and consequently but one vanishing point. The centre of the picture is the vanishing point of lines perpendicular to the picture (6).

PROPOSITION IV.

17. The projection of a line which is parallel to the picture is parallel to its original.

Fig. 6.



Let *E F G H* be the plane of the picture, and *A B* an original line parallel to that plane, *O* the point of sight, and *A O B* the optic cone, producing, by its intersection with the picture, the projection *a b*. Now *A B* being parallel to the plane *E F G H*, if we suppose a plane passing through *A B* to be parallel to that plane, the triangle *O A B* will cut the parallel plane in *A B*; and when two parallel planes are cut by another plane, their common sections with it are parallel.—*Euclid*, Book XI. Prop. 16. Hence *a b* is parallel to *A B*.

18. Cor. 1. The projections of several lines parallel to each other, and to the picture, are parallel to one another. Thus *a b* and *c d* are parallel to one another, and to their originals *A B* and *C D*.

19. Cor. 2. The projection *a b c d* of any plane figure *A B C D*, parallel to the picture, is similar to its original. For, having drawn the diagonal *A C*, and its corresponding projection *a c*, the sides *a b*, *b c*, *a c* are parallel to their corresponding originals, *A B*, *B C*, *A C*; wherefore the angles at *a*, *b*, and *c* are equal to the corresponding angles *A*, *B*, and *C*; consequently the triangle *a b c* is similar to the triangle *A B C*. For the same reason, *a c d* is similar to *A C D*; hence the figure *a b c d* is similar to *A B C D*.

20. Cor. 3. In this case, *a b* : *A B* : : the distance of the picture : the distance between the point of sight and the plane of the original figure.

Let *O k K* be perpendicular to the parallel planes, cutting them in *k* and *K*; then

a b : *A B* : : *O a* : *O A*.
: : *O k* : *O K*.—*Euclid*, Book XI. Prop. 17;

but *k* is the centre of the picture, *O k* is its distance, and *O K* is the distance of the point of sight from the plane of the original figure.

PROPOSITION V.

21. The projection of a straight line is parallel to its director. In Fig. 5 (p. 7), $P Q$ is an original line, $p q$ is its projection, and $O K$ its director. Since the directing plane $O K L$, and the picture $F G$ are parallel planes, and are cut by the plane of the triangle $O K Q$, their intersections $O K$ and $p q$ are parallel to one another.—*Euclid*, Book XI. Prop. 16.

22. Cor. 1. The projections of straight lines which have the same director are parallel to each other.

23. Cor. 2. When an original line is parallel to the picture, its directing point is at an infinite distance; its director is therefore parallel to it, and consequently is in the parallel of any plane passing through the original line. Whence, the vanishing line of that plane and the projection of the line are parallel to one another.

PROPOSITION VI.

24. The vanishing line, the intersection, and the directing line of any original plane are parallel to one another.

For the planes $R S T U$ and $L D M$ (the same figure being referred to) are parallel, as also the planes $O K L$ and $G H F$; therefore, the vanishing line $U T$, the intersection $G H$, and the directing line $D L$, are parallel to each other.—*Euclid*, Book XI. Prop. 16.

Cor. $V q : V I :: O V : K Q$. For the figure $O V I K$ is a parallelogram, whence $I V = O K$. Also the triangles $O V q$ and $O K Q$ are similar, therefore,

$$V q : O V :: O K \text{ or } V I : K Q;$$

alternately,

$$V q : V I :: O V : K Q.$$

PROPOSITION VII.

25. The vanishing points of all straight lines in any original plane are in the vanishing line of that plane.

Since all the original lines are in the same plane, their parallels which pass through the point of sight will be in the parallel plane; whence all the vanishing points are in the vanishing line of the original plane.

26. Cor. 1. Original planes that are parallel have the same vanishing line.

27. Cor. 2. Since the vanishing point of any straight line is in the vanishing line of the plane which the line is in, it follows that the vanishing point of the common section of two original planes (which section is a line in both planes) is the intersection of their vanishing lines.

28. Cor. 3. The vanishing line of a plane perpendicular to the picture passes through the centre of the picture (10).

PROPOSITION VIII.

29. The intersections of all straight lines in any original plane are in the intersection of that plane.

For all straight lines in any original plane *L D M*, Fig. 5, inclined to the picture as *P Q*, manifestly meet it in the line *G H*.

30. Cor. 1. The intersection of the common section of any two original planes is the point of intersection of their intersections.

Let *O P Q* and *L D M* be two original planes; then *P Q* is their common section, and the point *I* the intersection of that line.

But *I q* and *G H* are the intersections of those original planes, and intersect one another in the same point *I*.

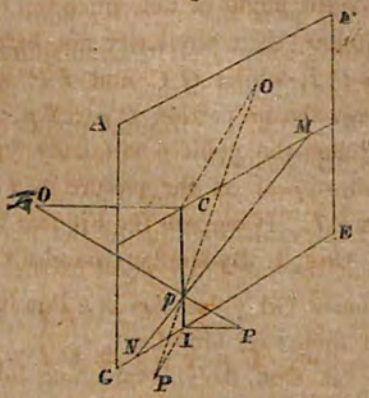
31. Cor. Planes whose common section is parallel to the picture have parallel intersections, and also parallel vanishing lines.

PROBLEM I.

Having given the centre and distance of the picture, also the seat* of a point, with its distance from the picture, to find the projection of that point.

Let O be the place of the eye of the spectator, $A\ G\ E\ F$ the picture, C its centre, P the point whose seat I on the picture is given and whose projection is required; then $O\ C$ is the distance of the picture, and $I\ P$ the distance of the point or object P from the picture. Join $O\ P$ and it will intersect the plane of the picture in p, the projection required.

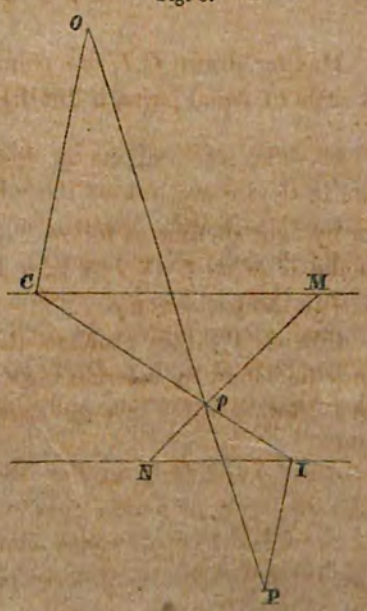
Fig. 7.



In practice we make use only of this plane, and obtain the point p as follows:—

In the plane of the picture and from its centre C draw at pleasure $C\ O$, Fig. 8, equal to the distance of the picture, and parallel to it draw $I\ P$ equal to the distance of the original point from its seat. Join $C\ I$ and $O\ P$, meeting each other in the point p, which is the projection sought.

Fig. 8.



If we suppose the angle $O\ C\ I$, and consequently the angle $P\ I\ p$, to be a right angle, then, turning the triangles $O\ C\ p$ and $P\ I\ p$

* When a point or a straight line is projected upon the picture by rays perpendicular to it, such projection is called the *seat* of that point or line.

round the line $C\ p\ I$, as an axis, till $C\ O$ and $I\ P$ become perpendicular to the picture, the point O in front would be the point of sight, the point P behind the picture will be the original point or object, and $P\ O$ is the visual ray cutting the picture in p, which consequently will be the projection of the point P. But the position of the point p is the same whether the angle $O\ C\ I$ is a right angle or not, since the triangles $O\ C\ p, p\ I\ P$ are always similar, their similarity not being affected by altering the angle $O\ C\ I$, whilst $O\ C$ and $I\ P$ continue to be parallel; in which case the proportion $C\ p : I\ p :: O\ C : P\ I$ remains the same. Hence the point p will truly represent the projection of a point whose seat on the picture is I and whose distance from its seat is $P\ I$. (Compare the Figures 7 and 8.)

Cor. 1. By similar triangles $O\ C\ p, p\ I\ P$,

$$C\ p : p\ I :: O\ C : I\ P;$$

$$\text{or,}\quad C\ p + p\ I = C\ I : p\ I :: C\ O + I\ P : I\ P;$$

$$\therefore p\ I = \frac{C\ I \,.\, I\ P}{C\ O + I\ P}.$$

Having drawn $C\ I$, the point p may be found by taking from a scale of equal parts, a line $I\ p$

$$= \frac{C\ I \,.\, P\ I}{C\ O + P\ I};$$

or by constructing as follows:—Draw $C\ M$, $I\ N$ horizontal, and make $C\ M = C\ O$, $I\ N = I\ P$, then joining $M\ N$ it will cut $C\ I$ in the projection p.

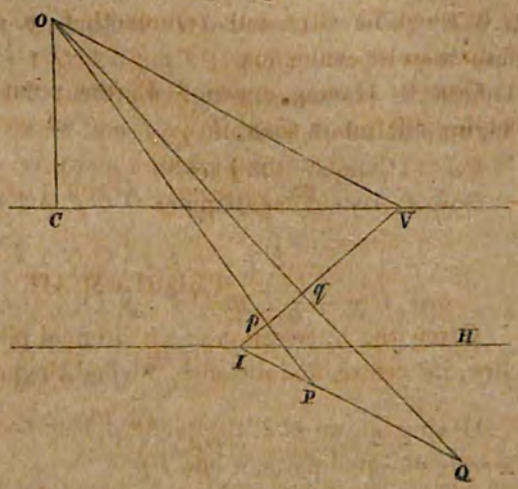
Cor. 2. By this problem the projection of any straight line may be found by determining the projections of two points in it and drawing a line through those projections.

PROBLEM II.

To find the projection of a line, also the vanishing point and its distance, having given its seat, its intersection, and the angle it makes with its seat, together with the centre and distance of the picture.

Let $I\,H$ be the given seat of the line proposed, I its intersection, and C the centre of the picture. Draw $I\,Q$, making the angle $H\,I\,Q$ equal to the angle which the original line makes with its seat, and through the centre C draw $C\,V$ parallel to $I\,H$, and $C\,O$ perpendicular to it, or to $C\,V$, and equal to the distance of the picture. Draw $O\,V$ parallel to $Q\,I$, meeting $C\,V$ in V, and join $I\,V$. Then V will be the vanishing point, $O\,V$ the distance of the vanishing point, and $I\,V$ the indefinite representation of the proposed line.

Fig. 9.



Imagine the planes $O\,C\,V$, $Q\,I\,H$ to revolve about the lines $C\,V$ and $I\,H$ as axes, till $O\,C$ and $Q\,H$ become perpendicular to the picture. Then O will be the point of sight, and $Q\,I$ the original line, and $O\,V$ being parallel to $Q\,I$, V is its vanishing point (11), and consequently, by Prop. 3, $I\,V$ is the projection.

Cor. 1. The projection of any part of the line, as $Q\,P$, may be found by drawing the lines $P\,O$ and $Q\,O$ as visual rays cutting $I\,V$ in p and q. For the line $Q\,P\,I$ being conceived to be the

original line brought into the plane of the picture by turning the plane $Q\,I\,H$ about the line $I\,H$ as an axis, along with triangle $O\,C\,V$ about $C\,V$, and $O\,V$ being constantly parallel to $I\,Q$, the position of the points p and q will not be affected, since the triangles $p\,V\,O$ and $p\,I\,P$, as also $q\,V\,O$ and $q\,I\,Q$, are still similar, and the proportions $p\,V : p\,I :: O\,V : I\,P$, and $q\,V : I\,q :: O\,V : I\,Q$, are not altered. To understand this more clearly, the reader may compare this figure with the figure at p. 3, where he will see the points O, V, q, p, I, P, Q, H, in both figures to be analogous.

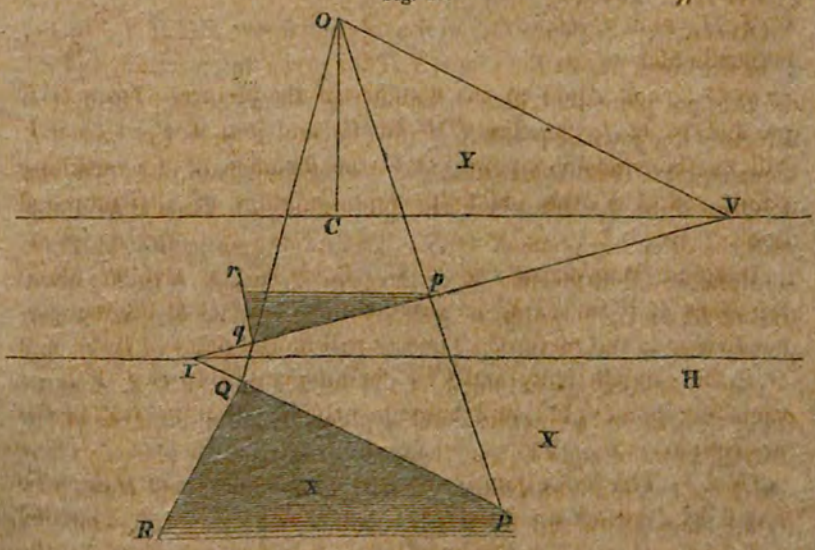
Cor. 2. Having drawn $I\,V$, the point q may be found by laying off from a scale, $V\,q$

$$= \frac{V\,I \cdot O\,V}{O\,V + I\,Q}. \text{ (Prob. I., Cor. 1.)}$$

PROBLEM III.

Given the intersection of an original plane, with its vanishing line, its centre, and distance, to find the projection of any line in

Fig. 10.



the original plane, having the original figures drawn out in their just proportions and placed in their proper position with regard to the picture

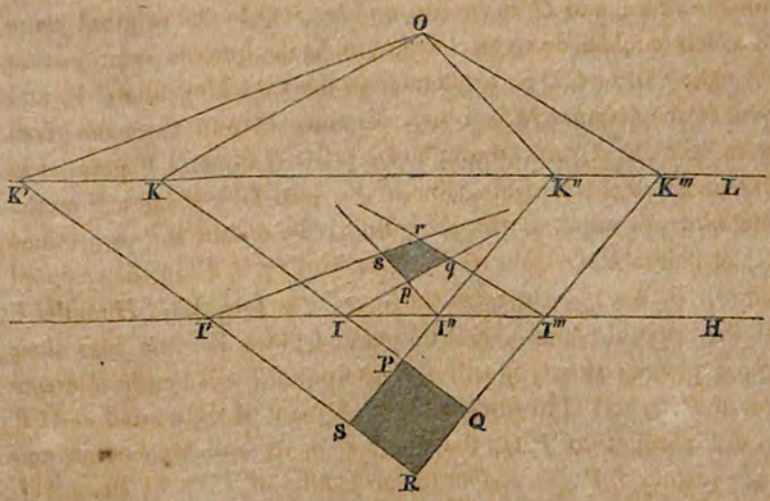
Let $I H$, Fig. 10, be the given intersection of the plane, $C V$ the vanishing line, and C its centre; and let $P Q$ in the original plane X, which is folded down on the picture, be the line whose projection is sought. Draw $C O$ perpendicular to the vanishing line $C V$, and equal to the distance of that line. Produce $P Q$ to meet the given intersection in I, and through the point O draw $O V$ parallel to $P Q$, meeting the vanishing line in V; join $I V$, which will be the indefinite projection of the given line. To obtain the projections of the points P, Q, join $O P$, $O Q$, cutting $I V$ in the required points p, q. For if we imagine the figure to be folded in $I H$ and $C V$ till the original plane X and its parallel plane Y come into their proper places; then you will find the point I will be the intersection of $P Q$, and O in front will be the point of sight; and as $O V$ is still parallel to $P Q$, the point V is its vanishing point, and consequently $I V$ the indefinite projection of $P Q$, by Prop. III. In the same way we obtain $q r$, the projection of another original line $Q R$ in the same plane.

Otherwise,—Suppose $P Q$ in the original figure $P Q R S$, Fig. 11, to be the given line in the plane, $I H$ the given intersection, and let the original plane be folded down on the picture so as to bring the directing line into the place $K L$, the distance between $I H$ and $K L$ being equal to the distance of the vanishing line given. Let O be the point of sight, brought into the picture at the same time along with the directing plane $K O L$. To find the projection of $P Q$, continue it till it meets $I H$ in the point I, and $K L$ in K; then join $O K$, and $I p q$ drawn parallel to it will be the indefinite projection sought.

Let the figure be unfolded in the lines $I H$ and $K L$ so as to bring the original plane, its parallel plane, and the directing plane with the point of sight O, back again into their proper places. Then will I, as before, be the intersection of $P Q$, and K will come into its directing point, and $I p q$ will still be parallel to $O K$, which is

the director of $P\,Q$; wherefore $I\,q$ is the indefinite projection of $P\,Q$ by Prop. V.

Fig. 11.



In the same manner are found the other lines $r\,s$, $p\,s$, $q\,r$, completing the projection $p\,q\,r\,s$ of the original figure $P\,Q\,R\,S$; the intersections being I', I'', I''', and the directing points brought into the plane of the picture, K', K'', K'''.

This construction is indeed only a particular case of the former; for if the parallel to $P\,Q$ be drawn, and the projection $I\,p\,q$ be produced, they will meet $K\,L$ in the vanishing point, whose distance will be equal to $K\,I$, agreeably to Art. 11.

32. The reader, by referring to Fig. 5, may conceive the manner of bringing the original plane $L\,D\,M$, its parallel plane $R\,T$, and the directing plane $D\,E$, with the point of sight O, into the plane of the picture, by supposing the original plane to revolve about its intersection $G\,H$ as an axis, and conceiving those several planes to be brought into the plane of the picture, by enlarging the angles $O\,K\,I$, $O\,V\,I$, till the three planes lie flat on the picture, and the original figures are seen on the reverse side of this plane.

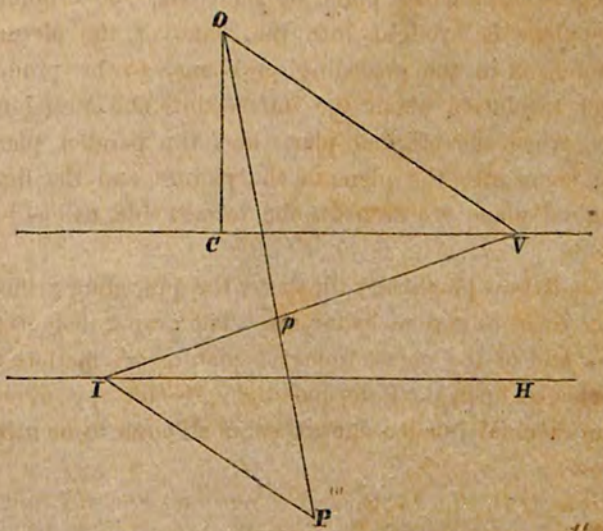
33. The vanishing line to the ground plane (supposed to coincide with the horizon and to be at right angles to the picture) is often called the *horizontal line*, and the intersection of the ground plane with the picture the *base* or *ground line*. When, in the following examples, the point of sight and the original point, line, or plane is brought into the plane of the picture, it is understood, as in the preceding problems, to be produced by (similar) revolution about the intersection and vanishing lines as axes, when the original plane and the parallel plane with the eye come into the plane of the picture, and the figures on the original plane are seen on the reverse side, as has just been remarked.

We shall now practically illustrate the preceding principles by applying them to various examples. The proper distances of the spectator and of the object from the picture are matters of some importance, as upon these depends the agreeable appearance of the representation. We shall assume them in all cases to be ascertained and given.

EXAMPLE 1.

To find the projection or representation of a given point.

Fig. 12.



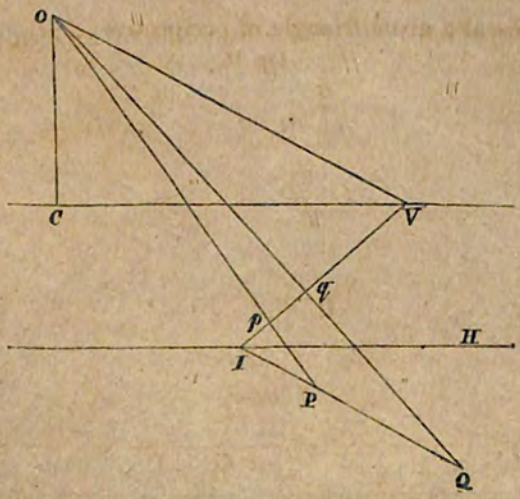
Let O be the point of sight, brought into the plane of the picture, C the centre of the picture, P the original point, whose projection is sought, also brought into the picture, and $C\,V$ and $I\,H$ the horizontal and ground lines. Draw any straight line $P\,I$, and through the point O draw $O\,V$ parallel to $P\,I$, meeting the horizontal line in the point V; join $I\,V$ and $O\,P$, intersecting each other in the point p, which is the representation required. Compare this with Fig. 5, p. 7.

EXAMPLE 2.

To find the representation of a given straight line.

Produce the given line $Q\,P$, Fig. 13, to the point I, and through the point O draw $O\,V$ parallel to $Q\,I$; then I will be the intersection and V the vanishing point of the given line. Join $I\,V$ and draw $O\,P$, $O\,Q$ cutting $I\,V$ in p, q; then $p\,q$ is the representation or projection sought, by Prop. III.

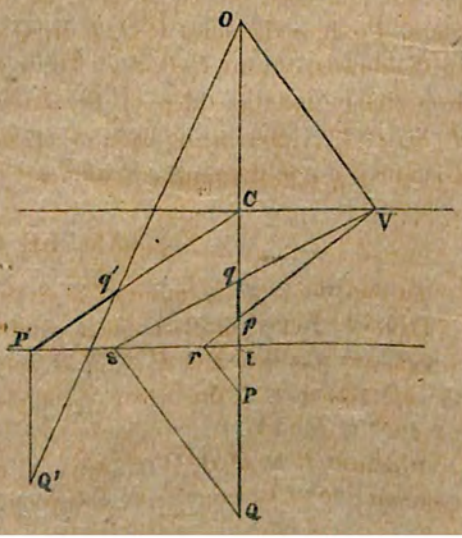
Fig. 13.



If the original line produced passes through the centre of the picture:—

Continue $Q\,P$, Fig. 14, to meet the ground line in I, and draw at pleasure the parallel straight lines $P\,r$, $Q\,s$, to meet the ground line in r, s; through O draw $O\,V$ parallel to $P\,r$ or $Q\,s$, and join $V\,r$, $V\,s$; then through C draw $C\,I$, cutting $V\,r$, $V\,s$ in p, q.

Fig. 14.

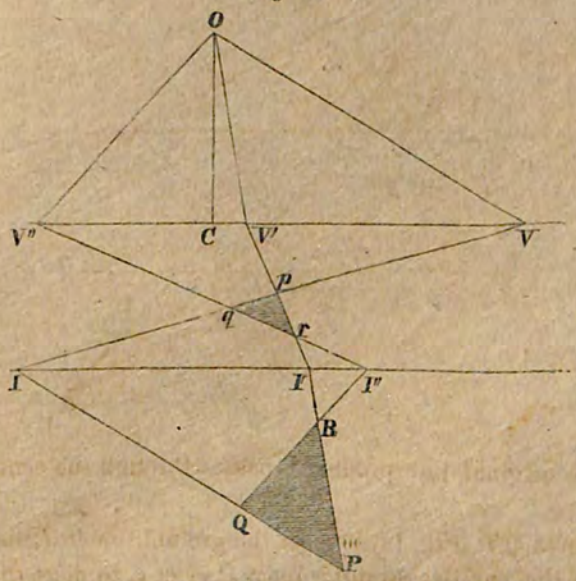


Suppose the line $P\,Q'$ to be perpendicular to the picture at the point P', then is C its vanishing point (16). Join $P'\,C$, $O\,Q$, intersecting in the point q'; $P'\,q'$ is the required representation in this case.

EXAMPLE 3.

To represent a given triangle in perspective.

Fig. 15.



Let $P\ Q\ R$ be the given triangle brought into the plane of the picture, and seen on the reverse, as objects appear in a looking-glass. Produce the sides $P\ Q$, $P\ R$, $Q\ R$, to meet the ground line in their intersections I, I', I''. Draw $O\ V$, $O\ V'$, $O\ V''$, parallel respectively to the sides of the triangle $P\ Q\ R$, and join $I\ V$, $I'\ V'$, $I''\ V''$, intersecting each other in the points p, q, r. The triangle $p\ q\ r$ is the representation sought.

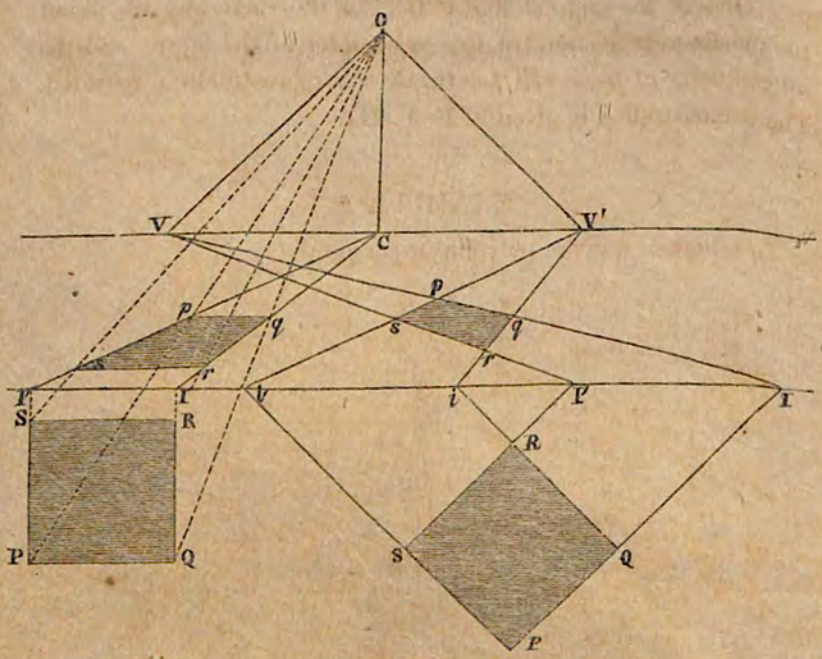
EXAMPLE 4.

To find the representation of a given square or rectangle.

Case 1. Let $P\ Q\ R\ S$ be a square, the sides $P\ Q$, $S\ R$ of which are parallel, and $P\ S$, $Q\ R$ perpendicular, to the plane of the picture, then C, the centre of the picture, is the vanishing point of $P\ S$, $Q\ R$ (16).

Produce $P\ S$, $Q\ R$, Fig. 16, to meet the intersection or ground line of the figure in the points I, I'. Draw $I\ C$, $I'\ C$,

Fig. 16.



and, O being the point of sight brought into the picture; join $O\,P$, $O\,S$, $O\,Q$, $O\,R$, cutting $I'\,C$, $I\,C$ in the points p, s, q, r, which are the projections of the four corners of the square.

Case 2. When the sides of the square or rectangle are inclined to the plane of the picture,—

Find the intersections $I\,I'$, $i\,i'$ of the four sides $P\,Q$, $S\,R$, $Q\,R$, $P\,S$ of the figure, and their vanishing points V, V'; then join $V\,I$, $V\,I'$, $V'i$, $V\,i'$; the intersections p, q, r, s of these lines will determine the representation required.

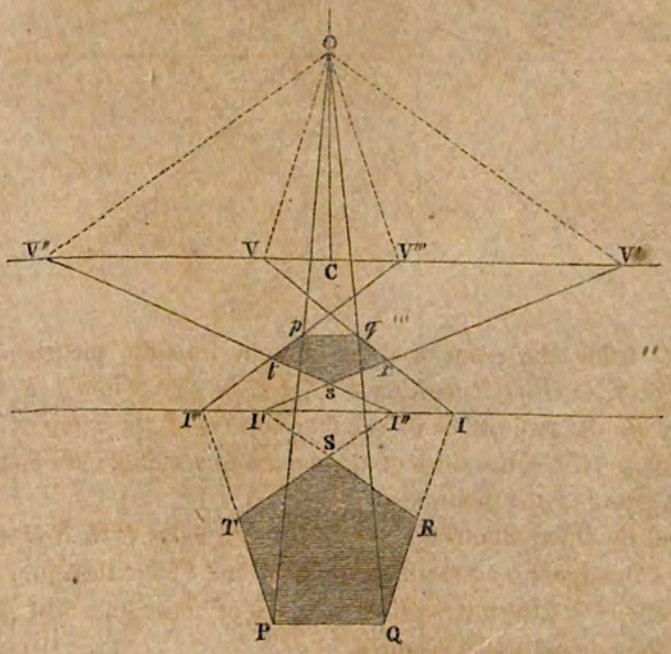
Otherwise,—Let the original figure be $P\,Q\,R\,S$, Fig. 11, situated in the plane whose intersection is $I\,H$; $K\,L$ the directing line, and O the point of sight, brought with the original figure and the directing plane into the plane of the picture;—the distance of the point of sight O from the directing line being, as in Prob. III., equal to the known distance of the vanishing line of the original plane $P\,Q\,R\,S$. Produce $P\,Q$ to its intersection I, and the directing point K, join $O\,K$, and through I draw

$I\,p\,q$ parallel to the director $O\,K$; then $I\,p\,q$ will be the indefinite projection of the original line $P\,Q$. In the same way are found the indefinite projections of the other sides of the figure, and the intersections of these will determine the representation required. The demonstration is given in Prob. III.

EXAMPLE 5.

To represent a given polygon in perspective.

Fig. 17.

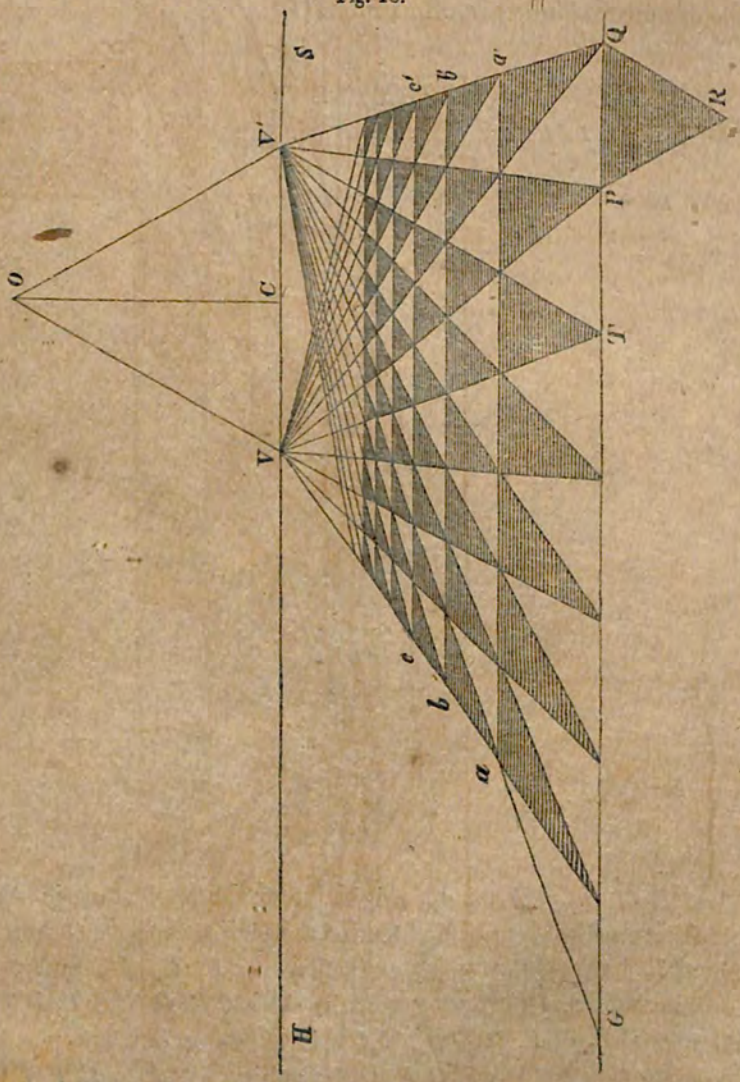


Let $P\,Q\,R\,S\,T$ be the original figure brought, as in the preceding examples, into the plane of the picture, and let O be the point of sight. Find the intersections I, I', I'', and the vanishing points, V, V', V'' V''', of the sides $Q\,R$, $R\,S$, $S\,T$, $P\,T$; and join $V\,I$, $V'\,I'$, $V''\,I'$, $V'''\,I'''$, intersecting one another in the points r, s, t. Draw $O\,P$, $O\,Q$, cutting $V'''\,I'''$ and $V\,I$ in p and q. Join $p\,q$, then will $p\,q\,r\,s\,t$ be the representation of the polygon $P\,Q\,R\,S\,T$.

EXAMPLE 6.

To represent a pavement in perspective.

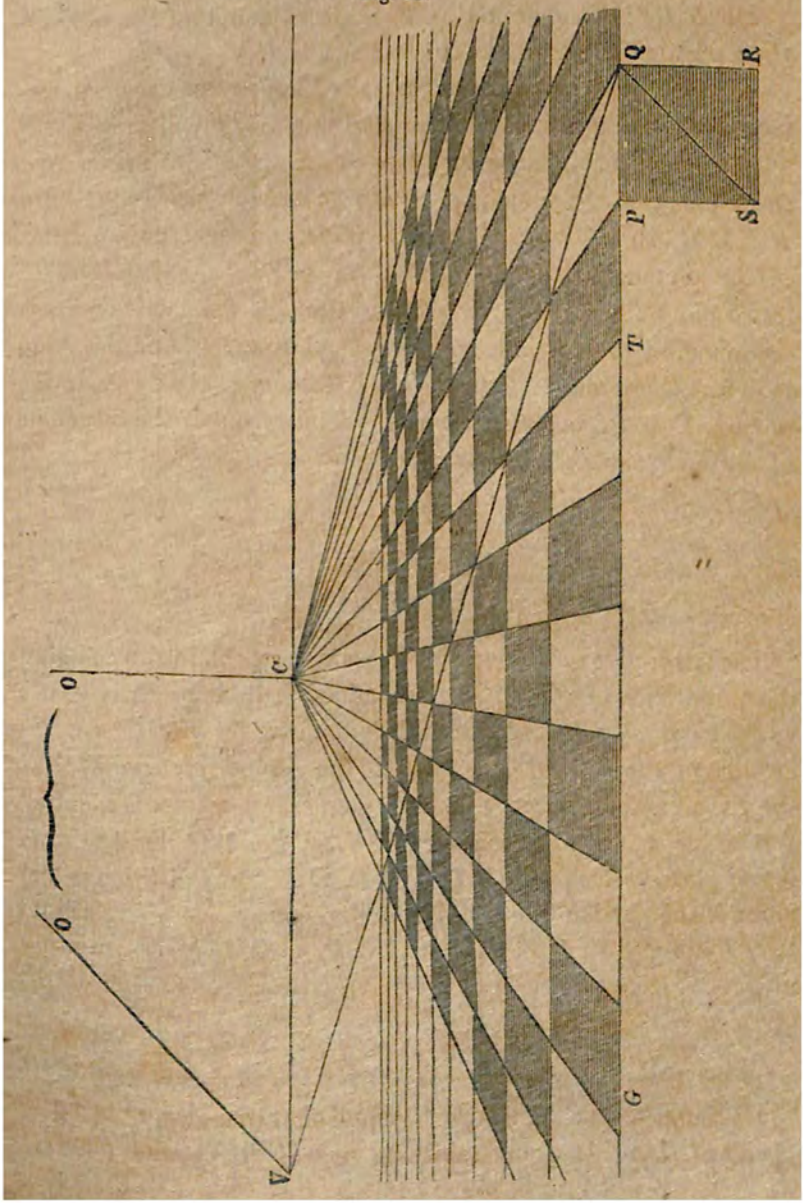
Fig. 18.



1. Suppose the figures to be equilateral triangles, O being the point of sight, $H\ S$ the vanishing line of the ground plane, or

the horizontal line, $G\,Q$ the ground line or intersection of the pavement with the picture, and $P\ Q\ R$ one of the original equilateral triangles. From the point O draw $O\ V$, $O\ V'$ parallel

Fig. 19.



respectively to $Q\,R$, $P\,R$; and along the ground line place distances $Q\,P$, $P\,T$, &c., equal to the side of the given equilateral triangle: join $Q\,V$, $Q\,V'$, $P\,V$, $P\,V'$, &c., and through the points of intersection draw $a\,a'$, $b\,b'$, &c., parallel to $G\,Q$, and it is done.

For $P\,R\,Q$ being an original figure, it is plain that V and V' are the vanishing points of the sides $Q\,R$ and $P\,R$.

2. Let the figures or the flagstones of the pavement be squares, as $P\,Q\,R\,S$, Fig. 19. On the ground line $G\,Q$ lay distances $Q\,P$, $P\,T$, &c., equal to the side of the given square, then C, the centre of the picture, is the vanishing point of the adjacent parallel sides, $P\,S$, $Q\,R$ (16). Draw $C\,V$ parallel to $G\,Q$, and equal to the distance of the picture; also draw $C\,Q$, $C\,P$, $C\,T$, &c., and join $Q\,V$, intersecting those converging lines; through the points of intersection draw lines parallel to the ground line $G\,Q$, and the thing is done. For since $C\,V = C\,O$, the distance of the eye from the picture, V is the common vanishing point of one of the diagonals of all the squares of the pavement *.

EXAMPLE 7.

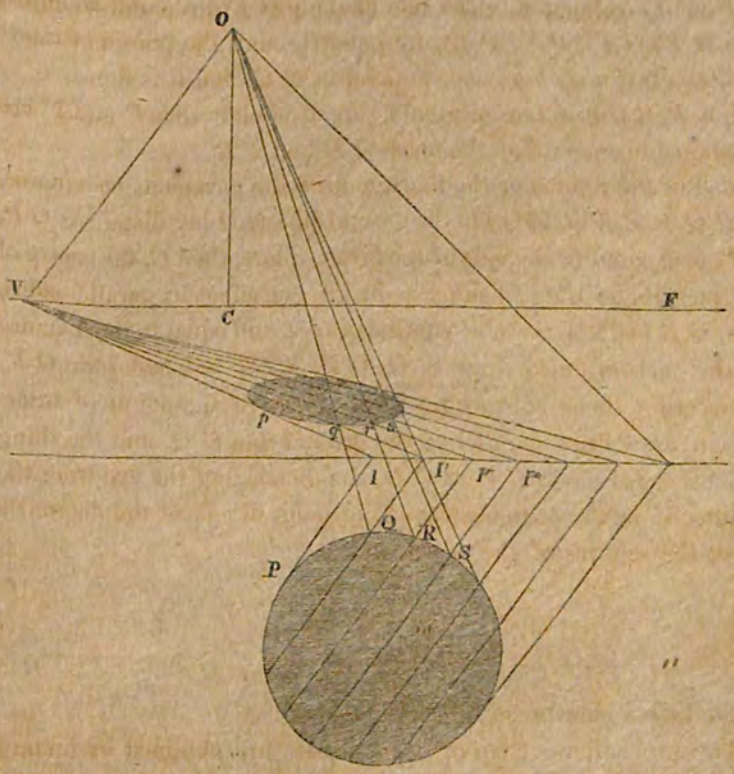
To draw a curvilineal figure in perspective.

The projections of curve lined figures are obtained by finding the projections of several of their points, and joining them neatly by the hand. Let O, Fig. 20, be the point of sight, $V\,F$ the vanishing line of the ground plane, $I\,I'\,I''$ the ground line, and $P\,Q\,R$ the given curvilineal figure, brought into the plane of the picture.

The projection of the points P, Q, R, &c. may be found by drawing at pleasure the parallels $P\,I$, $Q\,I'$, $R\,I''$, &c., and through the point O the parallel line $O\,V$; V is the common vanishing point: then joining $I\,V$, $I'\,V$, $I''\,V$, and $O\,P$, $O\,Q$, $O\,R$, &c., meeting in p, q, r, &c., the points of the perspective representation of the given curve.

* When the vanishing point is at the same distance from the centre of the picture as the point of sight, it is frequently called the *point of distance*.

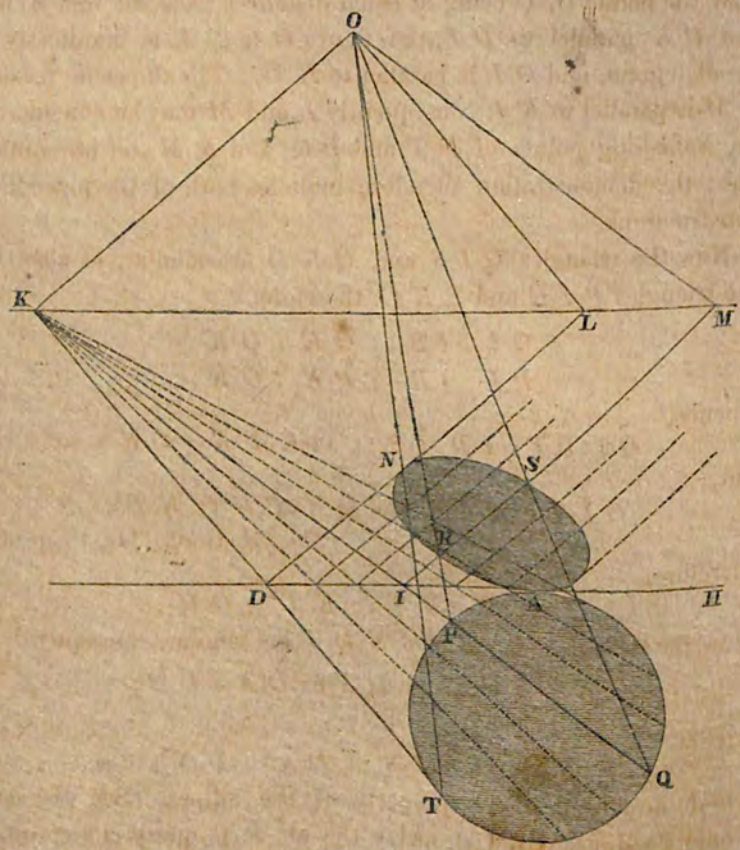
Fig. 20.



Otherwise,—Let *D H*, Fig. 21, be the intersection of the original plane of the figure, which is supposed to be folded down on the picture so as to bring the figure into the position *P Q T*, and the directing line into the position *K L*. Let *O* be the point of sight brought also into the plane of the picture, along with the directing plane *K O M*, the distance of the point of sight from the directing line *K L* being as before equal to the distance of the vanishing line of the plane of the original figure. Take any point *P* in the given curvilineal figure, and draw at pleasure the straight line *P I K*, cutting *D E* in the intersection *I*, and *K L* in the directing point *K*; then joining *O K*, draw *I M* parallel to the director *O K*, and join *O P* intersecting *I M* in the point *R*, which will be the projection of the original point *P*, by Prob. III. The same directing point

K is used for all the points Q, T, &c., and all the lines $D\,L$, $I\,M$, &c., being parallel to the same director $O\,K$ are parallel to each

Fig. 21.



other, and passing through the intersections D, I, &c., form the indefinite projections of the lines $D\,T$, $I\,P$, &c., according to Art. 22.

The perspective representation of a circle inclined to the picture is an ellipse.

Let the curve $P\,Q\,T$ be a circle, and suppose it to touch the base or ground line at the point A, and let $K\,D\,T$ be drawn touching the circle in T; produce $K\,P$ to Q and draw $D\,L$ parallel to $O\,K$ or $I\,M$: join $O\,T$, $O\,Q$, intersecting $D\,L$ in N,

and $I\,M$ in S; the points N and S will be the projections of the original points T and Q.—Prob. III.

Since $D\,I\,M\,L$ is, by construction, a parallelogram, $D\,I = L\,M$. And the points O, D being at equal distances from the line $K\,M$ and $O\,K$ parallel to $D\,L$, the figure $O\,K\,D\,L$ is manifestly a parallelogram, and $O\,L$ is parallel to $K\,D$. For the same reason $O\,M$ is parallel to $K\,I$, consequently L and M may be considered the vanishing points of $D\,T$ and $P\,Q$, and $K\,M$ the horizontal line; this demonstration therefore includes both of the preceding constructions.

Now the triangles $Q\,I\,S$ and $Q\,K\,O$ are similar, as also the the triangles $P\,I\,R$ and $P\,K\,O$, therefore,

$$Q\,I : I\,S :: Q\,K : O\,K$$
$$P\,I : I\,R :: P\,K : O\,K;$$

whence,

$$Q\,I.P\,I : I\,R.I\,S :: Q\,K.P\,K : O\,K^2;$$

but,

$$Q\,I.I\,P = I\,A^2 \text{ and } Q\,K.P\,K = K\,T^2,$$

Euclid, Book III., Prop. 36.

therefore,

$$I\,A^2 : I\,R.I\,S :: K\,T^2 : O\,K^2.$$

Now the triangles $O\,K\,T$ and $N\,D\,T$ are similar, consequently,

$$K\,T : O\,K :: D\,T \text{ or } D\,A : D\,N;$$

hence,

$$I\,A^2 : I\,R.I\,S :: D\,A^2 : D\,N^2,$$

which is a well-known property of the ellipse. See Hymer's Conic Sections, Art. 153, where the above property is expressed as follows:—

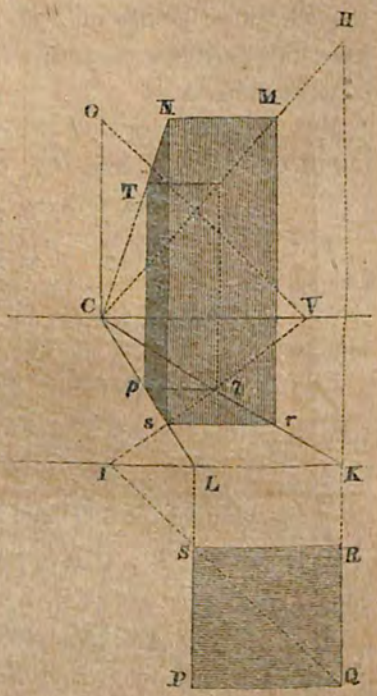
If from any point within or without an ellipse two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to one another in an invariable ratio. When the original circle $P\,Q\,T$ is given in magnitude, also its position, together with the point of sight and the centre of the picture or vanishing line, the straight lines $D\,A$ and $D\,N$ are also given in position and magnitude, and the ratio $D\,A^2 : D\,N^2$ is invariable.

EXAMPLE 8.

To find the projection of a cube or parallelopiped, having one of its sides parallel to the picture.

Let $P\ Q\ R\ S$ be the base of the solid brought into the plane of the picture, C the centre, and O the point of sight. Produce $P\ S$, $Q\ R$, $Q\ S$, to meet the ground line in the points L, K, I; join $C\ L$, $C\ K$, and draw $O\ V$ parallel to $Q\ I$; also join $I\ V$ meeting $C\ L$ and $C\ K$ in the points s and q, through which draw $p\ q$ and $r\ s$ parallel to $L\ K$ or $C\ V$. Prolong $Q\ K$ to H, and make $K\ H$ equal to the height of the given solid, and join $H\ C$. From the points r, s, p, draw the vertical lines $r\ M$, $s\ N$, $p\ T$; through the point M draw $M\ N$ parallel to $L\ K$, and join $C\ N$ meeting $p\ T$ in T, and the projection of the solid is completed*.

Fig. 22.



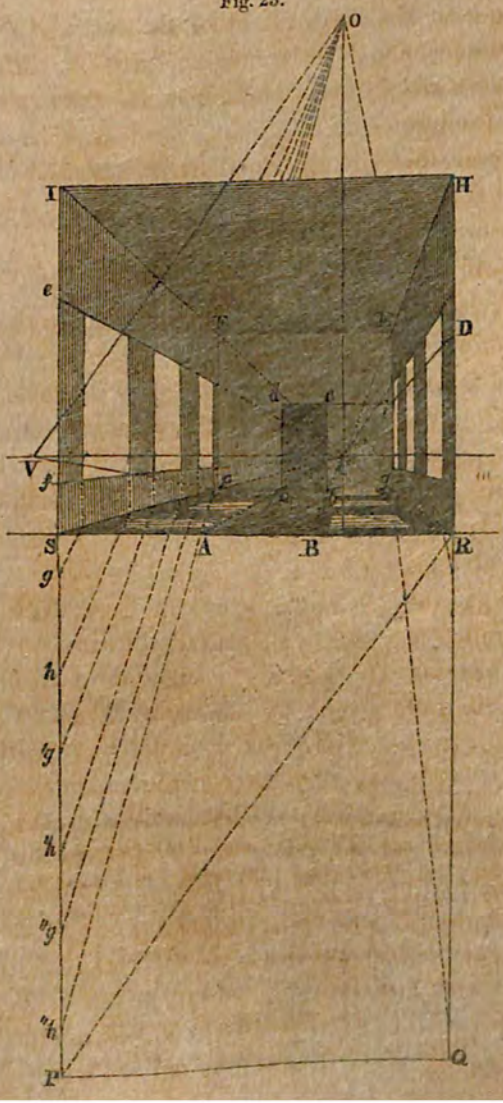
* The learner, if unable to follow the mathematical principles in the foregoing articles, may here test their accuracy by taking a square of glass and supporting it perpendicularly on a table with books; then placing a given cube at the distance of 2 or 3 inches from the pane of glass, and viewing it opposite to a given centre of the picture, at any convenient distance, say 8 inches. The picture of the cube as seen and measured upon the glass will exactly correspond to the above construction upon paper.

EXAMPLE 9.

To represent the interior of a room or building in perspective.

Let *P Q R S* be the ground plan or floor of the room, *C* the centre of the picture, *O C* the distance, and *O* the point of sight.

Fig. 23.



Draw the diagonal *P R*, and through the point *O* draw *O V* parallel to it, meeting the horizontal line, or vanishing line to the floor, in the point *V*, which will be the vanishing point of that diagonal. Join *S C* meeting *V R* in the point *p*; through *p* draw *p q* parallel to *S R*, meeting in *q* the line joining *R* and *C*. Produce *Q R* and *P S* to *H* and *I*, making *R H* and *S I* each equal to the height of the room from the same scale in which *R S* represents the breadth of the room, and *R Q* its length. Join *C H*, *C I* meeting in *E* and *F* verticals drawn from the points *q* and *p*, and draw *H I* and *E F*.

For the windows,—let $g\ h$, $g'\ h'$, &c., be their breadth, and $S\ e$, $S\ f$, the heights of the top and of the bottom of the windows. Join $O\ g$, $O\ h$, $O\ g'$, $O\ h'$, &c., intersecting $S\ p$ in points from which verticals are to be drawn to represent the sides of the windows. The projection of the top and bottom of the windows will be obtained by joining $C\ e$ and $C\ f$.

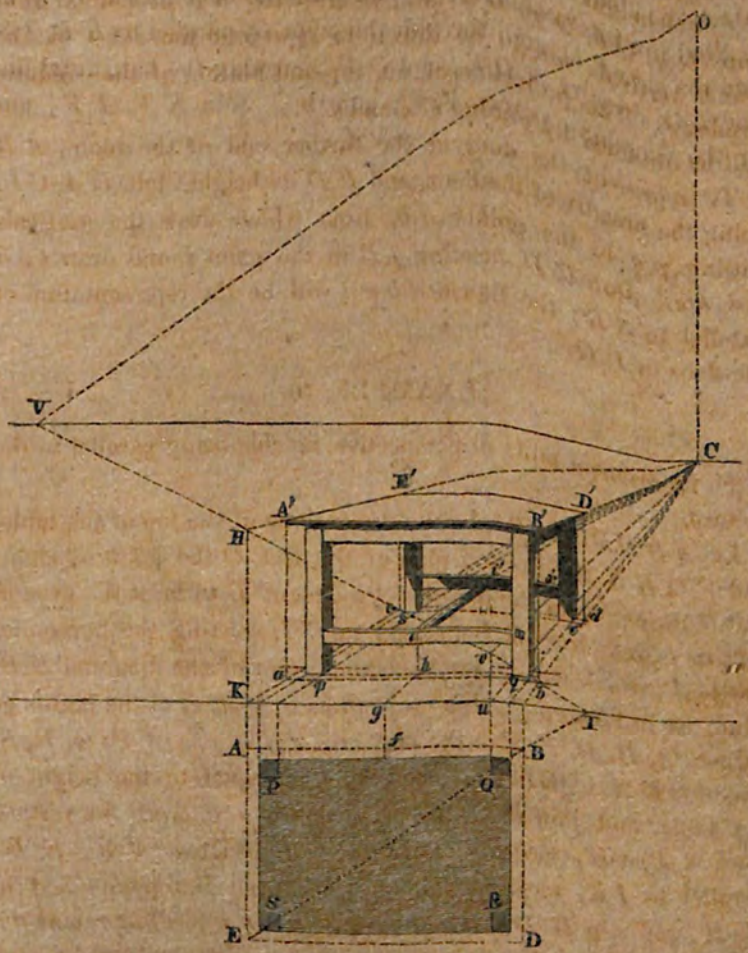
To represent the door at the further end of the room, $A\ B$ being the breadth of the door, and $R\ D$ its height: join $C\ A$, $C\ B$, cutting $p\ q$ in the points $a\ b$, from which draw the verticals $a\ d$, $b\ c$. Join $C\ D$, meeting $q\ E$ in the point i, and draw $i\ c\ d$ parallel to $S\ R$; the figure $a\ b\ c\ d$ will be the representation of the door in $P\ Q$.

EXAMPLE 10.

To represent a table in perspective, its side being parallel to the picture.

Let $A\ B\ D\ E$, Fig. 24, be the dimensions of the top of the table, and $P\ Q\ R\ S$ the ground plan of the feet, O the point of sight, and C the centre of the picture. Produce $E\ B$ to meet the ground line in I, and draw $O\ V$ parallel to $E\ I$, meeting the horizontal line in V, which will be the vanishing point of the diagonal $E\ B$. Find, as in Example 4, the projections a, b, d, e, of the points or corners A, B, D, E, also the projections p, q, r, s of P, Q, R, S. Produce $E\ A$ to K and H, making $K\ H$ equal to the height of the table, and join $H\ C$; from the points a, e, erect the vertical lines $a\ A'$, $e\ E'$, meeting $H\ C$ in A', E'. Draw $A'\ B'$, $E'\ D'$, parallel to $I\ K$, meeting vertical lines from the points b, d in B', D', and join $B'\ D'$. The figure $A'\ B'\ D'\ E'$ will represent the top of the table. The representation of the legs will be obtained by drawing vertical lines from the projections p, q, r, s of the feet. For the rails,—draw $Q\ u\ v$ perpendicular to $I\ K$, and making $u\ v$ equal to the height of the rails; join $v\ C$, meeting the legs of the table in x, x', and draw $x\ i$, $x'\ i'$ parallel to $I\ K$. Bisect $A\ B$ in f, draw $f\ g$ perpendicular to $I\ K$, join $g\ C$, cutting $p\ q$ in h, and draw $h\ i$ vertical, meeting $x\ i$ in i; lastly, draw $i\ i'\ C$, intersecting $x\ i'$ in i': the line i, i' is the centre of the middle rail of the table.

Fig. 24.



EXAMPLE 11.

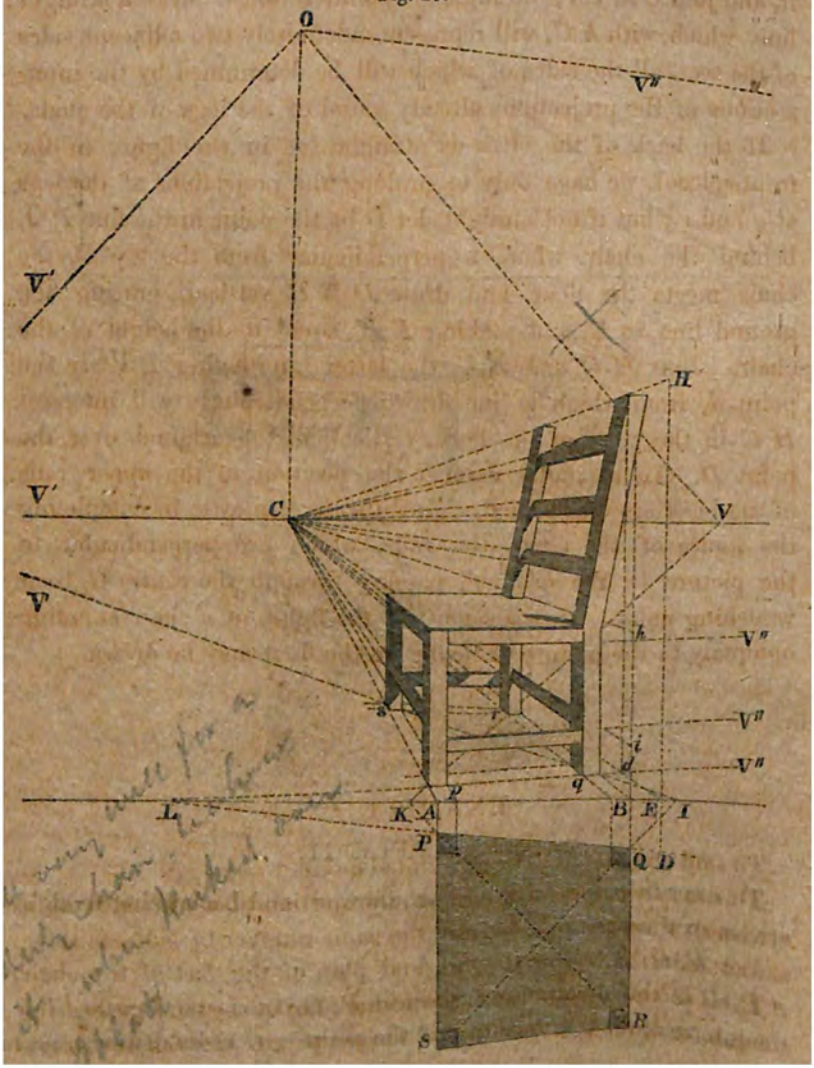
To draw in perspective a chair, its front and back being at right angles to the picture.

Let *P*, *Q*, *R*, *S* be the original plan of the feet of the chair, *A P*, *B Q* the distances of the feet *P*, *Q*, from the picture, *A B* the intersection, *C* the centre of the picture, *C O* its distance, and

V C V' the horizontal line or the vanishing line to the floor on which the chair stands.

Find the intersections *K*, *I* and *L* of the diagonals *P R*, *Q S*, and the side *P Q*, and, through the point *O*, parallel to *K R*, *I S*, and *L Q*, draw *O V*, *O V'*, and *O V''*, meeting the horizontal line in the vanishing points *V*, *V'*, and *V''*. Join *K V*, *I V'*, and

Fig. 25.



$L\ V''$, which will be the indefinite representations of the diagonals $P\ R$, $Q\ S$, and of the side $P\ Q$; and determine the projections p, q, r, s of the feet of the chair. From these projections draw lines vertically to represent the legs of the chair. Produce $R\ Q$ to the ground line at B, and prolong it, making $B\ h$ and $B\ i$ equal respectively to the height of the seat and of the back rail below it, and join $C\ h$, $C\ i$; through the points h and V'' draw a straight line, which, with $h\ C$, will represent indefinitely two adjacent sides of the seat, all the sides of which will be determined by the intersections of the projections already found of the legs of the chair.

If the back of the chair is straight (as in the figure in the frontispiece), we have only to prolong the projections of the legs at q and r; but if not straight, let D be the point in the line $P\ Q$, behind the chair, where a perpendicular from the top of the chair meets the floor, and draw $D\ E\ H$ vertical, cutting the ground line in E, and making $E\ H$ equal to the height of the chair. Join $H\ C$ and $E\ C$, the latter intersecting $L\ V''$ in the point d, from which a line drawn perpendicularly will intersect $H\ C$ in the point of the back, vertical, in the original, over the point D. In a similar manner the position of the upper rails of the back are determined. No difficulty can arise in completing the figure of the chair, the rails, which are perpendicular to the picture in the original, passing through the centre C, their vanishing point. In the same way the figure of a chair standing obliquely to the picture or laying on the floor may be drawn.

EXAMPLE 12.

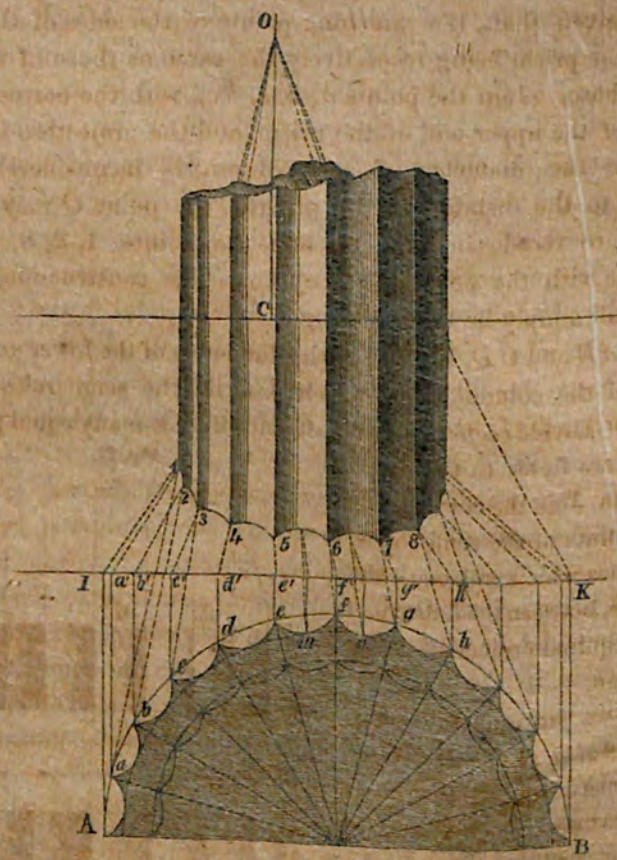
To find the perspective of a fluted column.

This may be done by finding the projection of a truncated right cylinder, or rather pyramid, of the same number of sides as there are flutes in the column.

Let $A\ a\ b\ c\ B$, Fig. 26, be the ground plan of the semi-base of the column, a, b, c, &c., the original visible points between the flutes,

brought into the plane of the picture, $I K$ the intersection, O the point of sight, C the centre of the picture, and consequently $O C$ is its distance.

Fig. 26.



From the points a, b, c, &c., draw $a a'$, $b b'$, $c c'$, &c., perpendicular to $I K$; join $C a'$, $C b'$, $C c'$, &c., and $O a$, $O b$, $O c$, &c., intersecting in the points 1, 2, 3, &c., which will be the projections of the original points a, b, c, &c.

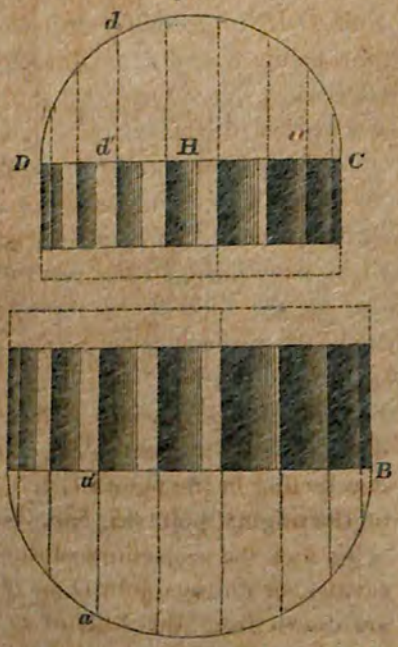
To find the projections of the points m, n, situated in the concavities, or flutings, join $O m$, $O n$, intersecting the lines which are drawn from the seats of m and n, to the centre C, in the required projections of those points.

In the same way find the projection of the horizontal section of the upper part of the shaft, placed concentrically within the figure *A a b c B*, and, considering it as a polygon, complete upon this base the projection of a right prism, whose altitude is equal to that of the given shaft, the vanishing points of the sides of the upper end of the prism being respectively the same as those of the sides of the base. Join the points *a*, *b*, *c*, &c., with the corresponding points of the upper end of the prism, and the projection is done.

When the diameter of the column is inconsiderable with respect to the distance of the picture, the point *O* may be considered to recede indefinitely, and the points, 1, 2, 3, &c., will coincide with the points *a'*, *b'*, *c'*, &c. The construction, in this case, will simply be the following.

Let *A B* and *C D*, Fig. 27, be the diameters of the lower and upper ends of the column, upon which describe the semicircles *A a B*, *C d D*. Divide each semi-circumference into as many equal parts as there are flutes in the semi-column. Join the corresponding points of the semicircles, by lines which will mark the ridges, between the flutings, of the required projection of the column.

Fig. 27.



If we imagine the semicircles *A a B* and *D d C* to be raised up, so that their planes may be at right angles to the picture, they may be considered as horizontal sections of the column, in which case the original points *a*, *d*, will be projected upon the same points or seats *a'*, *d'*, as by the construction *.

* The Doric, Ionic, Corinthian, and Composite columns are commonly fluted; the Tuscan column seldom.

EXAMPLE 13.

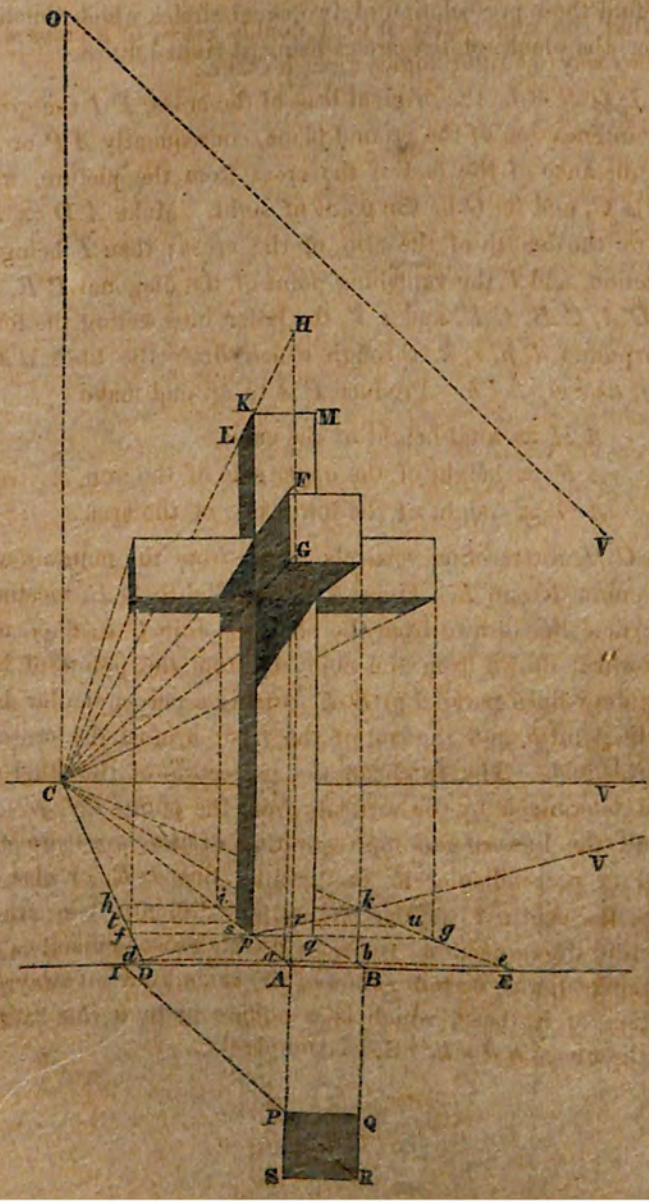
To find the representation of a double cross, or a cross which faces two ways at right angles to each other.

Let $P\ Q\ R\ S$ be the original base of the cross, $E\ I$ the ground line or intersection of the ground plane, consequently $A\ P$ or $B\ Q$ is the distance of the foot of the cross from the picture, whose centre is C, and let O be the point of sight. Make $A\ D = B\ E$ equal to the length of the arm of the cross; then I being the intersection, and V the vanishing point of the diagonal $P\ R$, join $C\ D$, $C\ A$, $C\ B$, $C\ E$, and $I\ V$, the latter intersecting the former in the points d, p, r, k, through which draw the lines $d\ a\ b\ e$, $f\ p\ q\ g$, $t\ s\ r\ u$, $h\ i\ k$. Produce $P\ A$ to H, and make

$A\ H =$ total height of the cross,
$A\ F =$ height of the upper side of the arm,
$A\ G =$ height of the lower side of the arm.

Join $C\ H$, intersecting verticals drawn from the points p and s in the points K and L. Draw $K\ M$ parallel to $D\ E$, meeting in M a vertical line drawn from the point q. Join $C\ F$, $C\ G$, meeting a vertical drawn from the point a; from the points of intersection draw lines parallel to $D\ E$, cutting a perpendicular drawn from the point b, and the end of the front arm of the cross will be represented. The extent of the projection of the other arms will be determined by the verticals from the points i, f, g. And since all the lines of the representation of the cross are either parallel or perpendicular to the ground line $D\ E$, or else pass through the centre C of the picture, no difficulty can arise in completing the projection. Indeed a double cross is merely a solid parallelopiped, with certain solids of the same kind cut away. The projection of its base, which is a square upon a line $= D\ E$, being the figure $h\ d\ e\ k$. See Example 8.

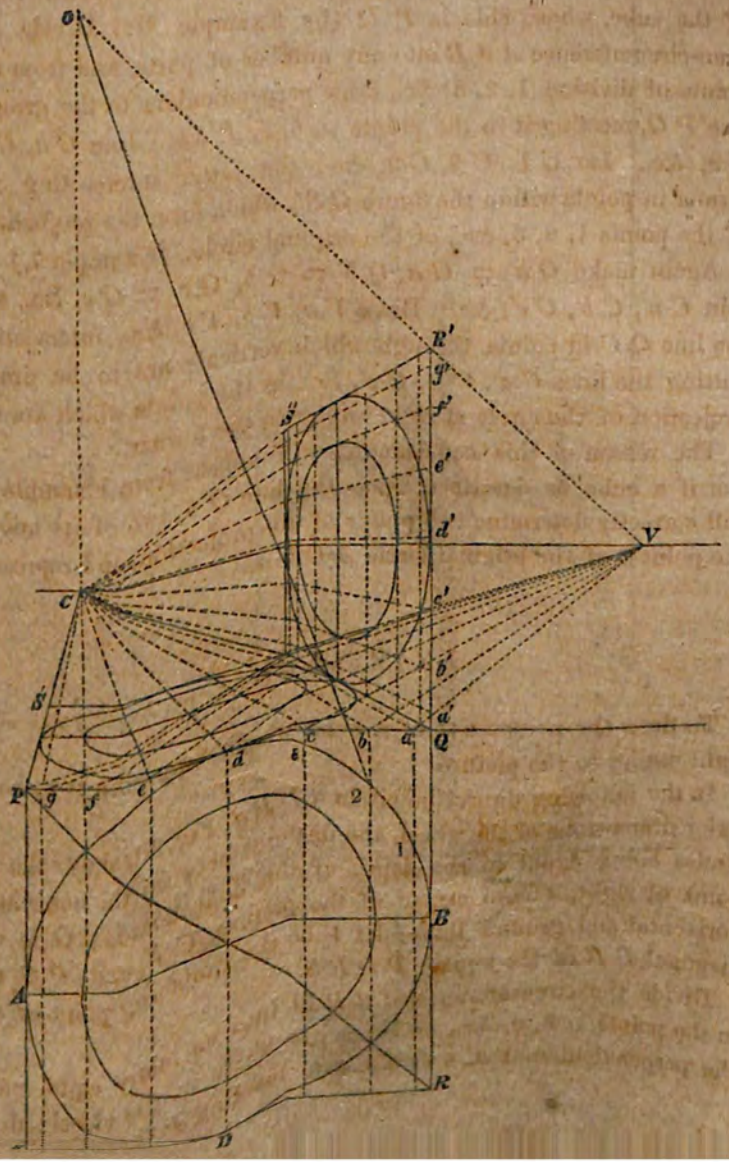
Fig. 28.



EXAMPLE 14.

To find the representation of two equal circles which touch one another, the planes of the circles being at right angles.

Fig. 29.



Let $A\ D\ B\ d$ be one of the given circles, $P\ Q\ R\ S$ its circumscribing square, O the point of sight, C the centre of the picture, $C\ V$ the horizontal line, $P\ Q$ the ground line, V the point of distance, or the vanishing point of the diagonal of the square $P\ R$. Find the representations $Q\ S'$, $Q\ S''$ of two adjacent faces of the cube, whose side is $P\ Q$ (by Example 8). Divide the semi-circumference $A\ d\ B$ into any number of parts, and from the points of division 1, 2, 3, &c., draw perpendiculars to the ground line $P\ Q$, meeting it in the points a, b, c, d, &c. Join $C\ a$, $C\ b$, $C\ c$, &c., also $C\ 1$, $C\ 2$, $C\ 3$, &c., the latter intersecting the former in points within the figure $Q\ S'$, which form the projections of the points 1, 2, 3, &c., of the original circle. (Example 7.)

Again make $Q\ a' = Q\ a$, $Q\ b' = Q\ b$, $Q\ c' = Q\ c$, &c., and join $C\ a'$, $C\ b'$, $C\ c'$, &c. Draw $V\ a$, $V\ b$, $V\ c$, &c., intersecting the line $Q\ C$ in points through which verticals are to be drawn cutting the lines $C\ a'$, $C\ b'$, $C\ c'$, &c., in the points which are the projection of the circle at right angles to the former.

The reason of this construction will appear from Example 8. For if a cube be described upon the line $Q\ b$, two of its angles will correctly determine two points of this projection, and represent the point 2 of the original circle $A\ D\ B\ d$.

EXAMPLE 15.

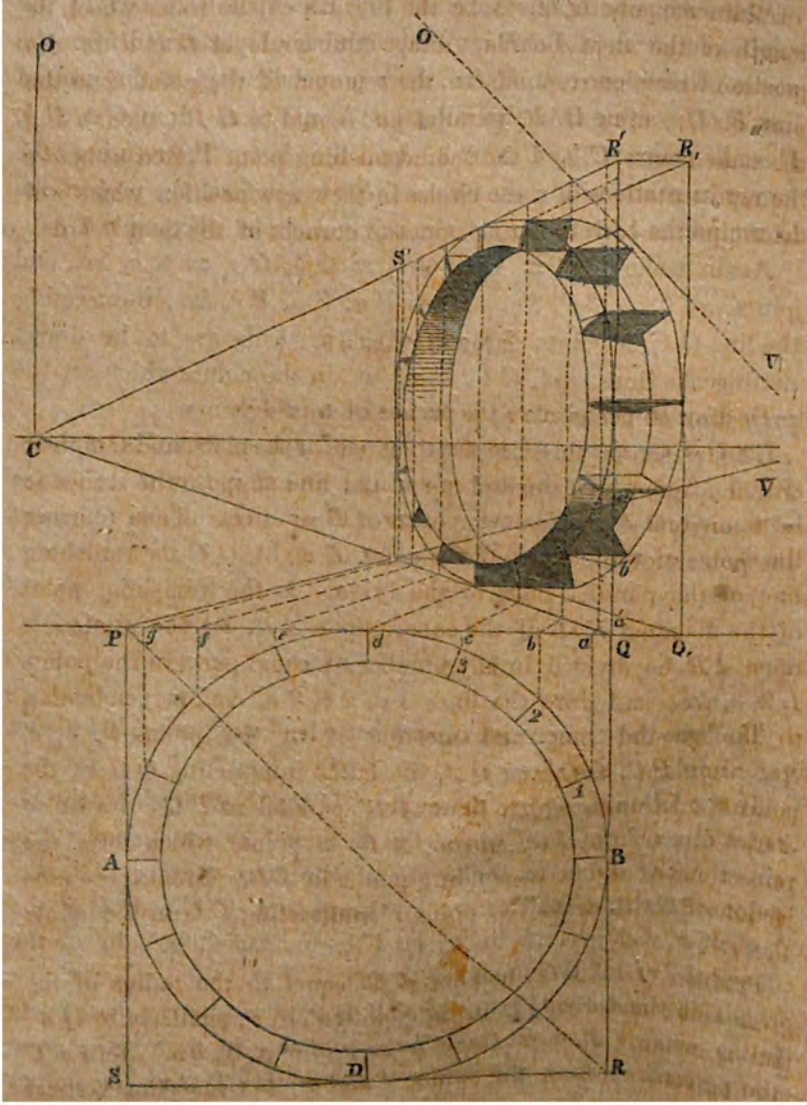
To draw the perspective of a water-wheel, supposing it to be at right angles to the picture.

In the following figure the circle $A\ D\ B\ d$, Fig. 30, shows the exterior dimensions of the wheel, the distance between the concentric circles being equal to the depth of the float boards; O is the point of sight, C the centre of the picture, $C\ V$ and $P\ Q$ the horizontal and ground lines; let V be the vanishing point of the diagonal $P\ R$ of the square $P\ Q\ R\ S$.

Divide the circumference $A\ d\ B\ D$ into as many equal parts in the points 1, 2, 3, &c., as there are floats in the wheel; draw the perpendiculars $1\ a$, $2\ b$, $3\ c$, &c., meeting $P\ Q$ in the points

a, *b*, *c*, &c. Produce *R Q* to *R'*, making *Q R'* equal to *R Q* or *P Q*; divide *Q R'* in the points *a'*, *b'*, *c'*, &c., similarly to *Q P*. Join *V a*, *V b*, *V c*, &c., cutting *Q C* in points from which verticals or parallels to *Q R'* are to be drawn. Also draw *C a'*, *C b'*, *C c'*,

Fig. 30.



&c., intersecting those vertical lines in points which are the projections of the exterior corners of the float boards.

By dividing similarly the circumference of the inner circle, we obtain by the same construction the projections of the interior corners of the floats.

Again suppose $Q\,Q_1$ to be the breadth of the water wheel, or length of the float boards. Place the circle $A\,D\,B\,d$ in the position which corresponds to the removal of the point Q to the position Q_1; draw $Q_1\,R_1$ parallel and equal to $Q\,R$; then, using the same centre C, and the same vanishing point V, find as before the representations of these circles in their new position, which will determine the corresponding opposite corners of the float boards.

EXAMPLE 16.

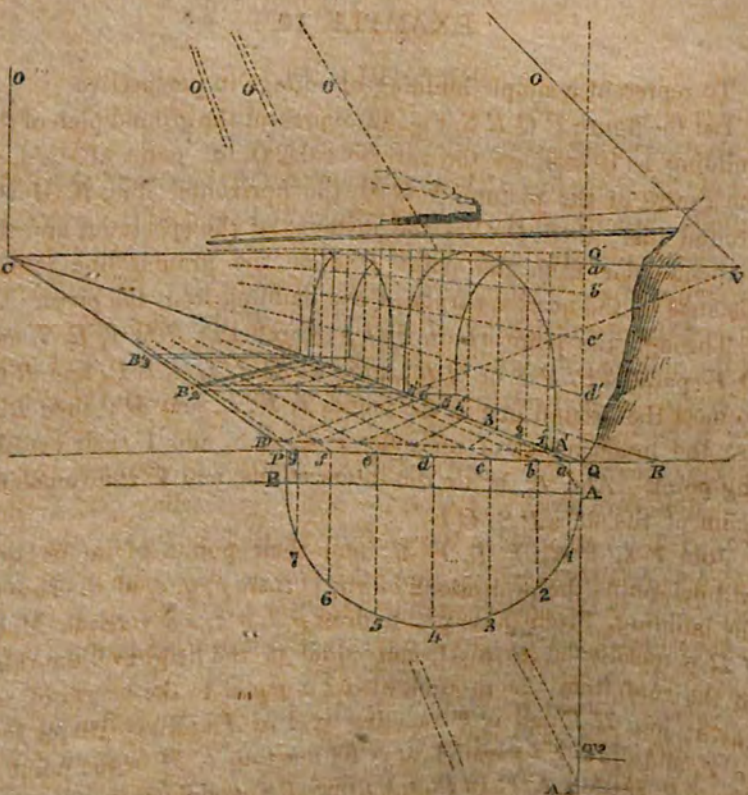
To draw in perspective the arches of a bridge.

Let $Q\,A, Q_2\,A_2$, Fig. 31, be the thickness of the piers, and $Q\,d'$ their altitude, from which the arches spring; and suppose the arches to be semicircular, the diameter being $A\,B$ or $A\,Q_2$. Then C being the centre of the picture, O the point of sight, $O\,C$ the vanishing line of the parallel sides of the bridge, V the vanishing point of the diagonal $B\,Q_2$ of the square upon $A\,B$, let the semicircle upon $A\,B$ be divided in any number of equal parts in the points 1, 2, 3, &c., and draw the lines $1\,a$, $2\,b$, $3\,c$, &c., perpendicular, to the ground line, and meeting it in the points a, b, c, &c. Join $P\,C$, $Q\,C$, and $O\,A$, the latter intersecting $Q\,C$ in the point A'; through which draw $A'\,B'$ parallel to $P\,Q$; draw also $a\,C$, $b\,C$, $c\,C$, $d\,C$, &c., cutting $A'\,B'$ in points which form the projections of the corresponding points in $A\,B$. From these projections draw lines to the point V intersecting $C\,Q$ in the points $1', 2', 3', 4'$, &c.

Produce $Q\,d'$ to Q', making $d'\,Q'$ equal to the radius of the given arch; and divide it in the points a', b', c', similarly to $Q\,d$; that is, making $d'\,c' = d\,c$, $c'\,b' = c\,b$, $b'\,a' = b\,a$. Join $c'\,C$ meeting verticals from the points $1'$ and $7'$; $b'\,C$, meeting vertical

lines from 2′ and 6′; and a' C, meeting perpendiculars from the points 3′ and 5′; these intersecting points will determine six points in the projection of the first arch of the bridge. The projection of the crown of the arch will be determined by the intersection of Q' C with a perpendicular drawn from the point 4′: the arch springs from the edge of the pier vertical over the point A'. The projection of the face A' d' of the first pier is already determined. For the second pier join P V, B' V, which will intersect O C in two points representing the thickness of the second pier. Or the same points will be obtained by drawing O A_2, O Q_2, cutting

Fig. 31.



$C\,Q$ in those points, through which draw lines parallel to $P\,Q$, $A'B'$; and the six points in the projection of the second arch will be obtained by using these parallels instead of $P\,Q$ and $B'A'$, and employing the same lines $a\,C$, $b\,C$, $c\,C$, &c., $c'\,C$, $b'\,C$, $a'\,C$, and the same vanishing point V. The crown of the arch will be in the line $C\,Q'$, its base springing from the piers in the line $C\,d'$.

In a similar way the third arch is drawn, and so on.

For the opposite and parallel side of the arches, we may make $Q\,R$ equal to the breadth of the bridge and proceed as directed in the corresponding case of the wheel in the last Example.

EXAMPLE 17.

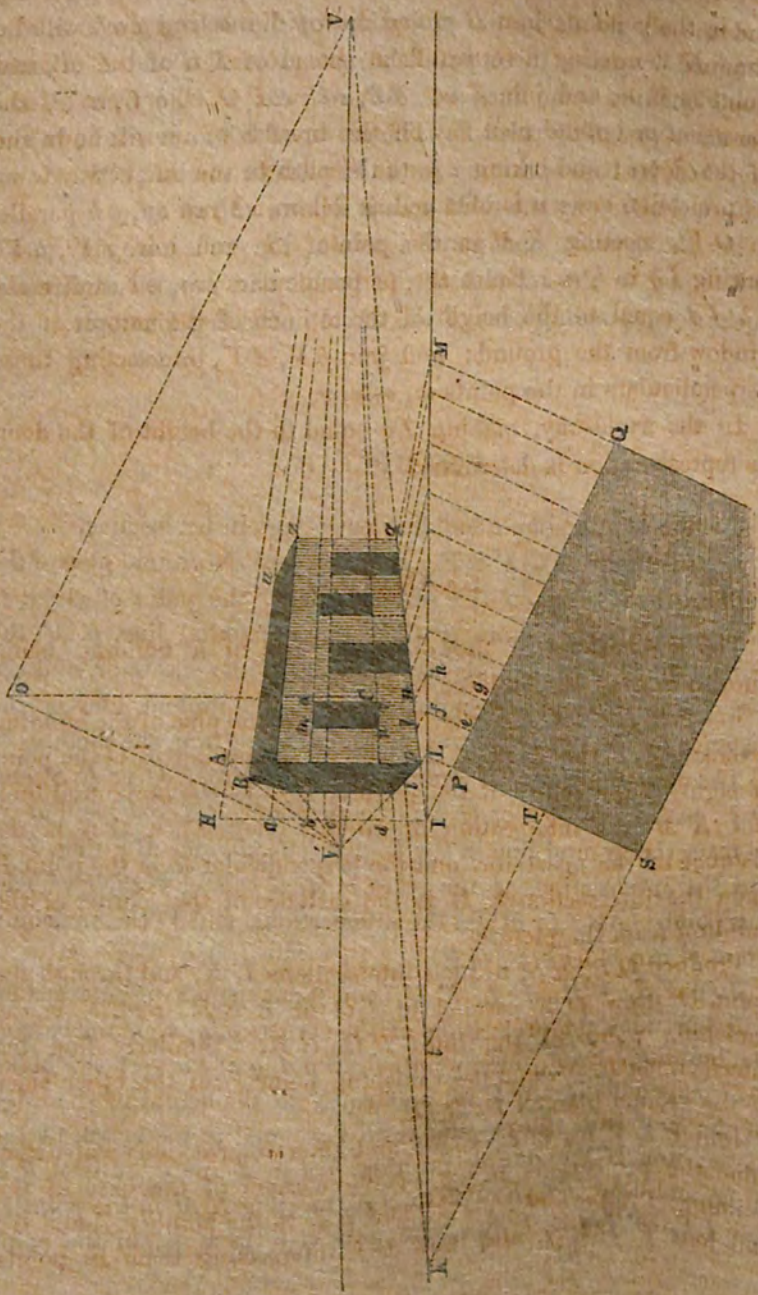
To represent a simple building or cottage in perspective.

Let the figure $P\,Q\,R\,S$, Fig. 32, represent the ground plan of the building as if seen on the reverse side, O the point of sight, C the centre of the picture, $V\,C\,V'$ the horizontal line, $K\,M$ the ground line; then $O\,C$ is the distance of the spectator, and the perpendicular from the point P upon the ground line is the distance of the nearest corner of the building from the picture.

Through the point O draw $O\,V$, parallel to $P\,Q$ or $R\,S$, and $O\,V'$ parallel to $P\,S$ or $Q\,R$; produce $Q\,P$, $R\,S$, $S\,P$, and $R\,Q$, to meet the ground line in the points I, K, L, and M; then I, K are the intersections of the sides $P\,Q$, $R\,S$, and V their vanishing point. Also L, M are the intersections, and V' the vanishing point of the sides $P\,S$, $Q\,R$.

Join $V\,I$, $V\,K$, $V'\,L$, $V'\,M$, and their points of intersection will determine the projections of the corners p, q, s, of the base of the building. From these points draw $p\,A$, $q\,z$, $s\,x$ vertical. Make $I\,H$ perpendicular to $K\,M$, and equal to the height of the ridge of the roof from the ground, also $I\,a$ equal to the height of the eaves; join $H\,V$ and $a\,V$ meeting $p\,A$ in A and y. Bisect $P\,S$ in T, and draw $T\,t$ parallel to $S\,K$, meeting $K\,M$ in the point t; join $t\,V$, cutting $L\,V'$ in i, and draw $A\,V'$ meeting, in the point

Fig. 32.



B, a vertical line drawn from the point i. Draw $y\ V'$ intersecting $s\,x$ in the point x; join $B\,x$, and draw $y\ V$ meeting $q\ z$ in z, also draw $H\ V$ meeting a vertical from the projection of the original point R in w, and join $Z\ w$. Again, in $P\,Q$, the front of the basement or ground plan, lay off the breadth of the windows and of the doors; and taking $e\,g$, the breadth of one of the windows, its projection $m\ o\ v\ u$ is obtained as follows: Draw $e\,f$, $g\,h$ parallel to $O\ V'$, meeting $K\,M$ in the points f, h, and join $f\ V'$, $h\ V'$, cutting $I\,q$ in l, n. Erect the perpendiculars $l\,m$, $n\ o$, and make $I\,b$, $I\,d$ equal to the height of the top and of the bottom of the window from the ground; and join $b\ V$, $d\ V$, intersecting those perpendiculars in the points m, o, u, v.

In the same way, making $I\,c$ equal to the height of the door, its representation is determined.

EXAMPLE 18.

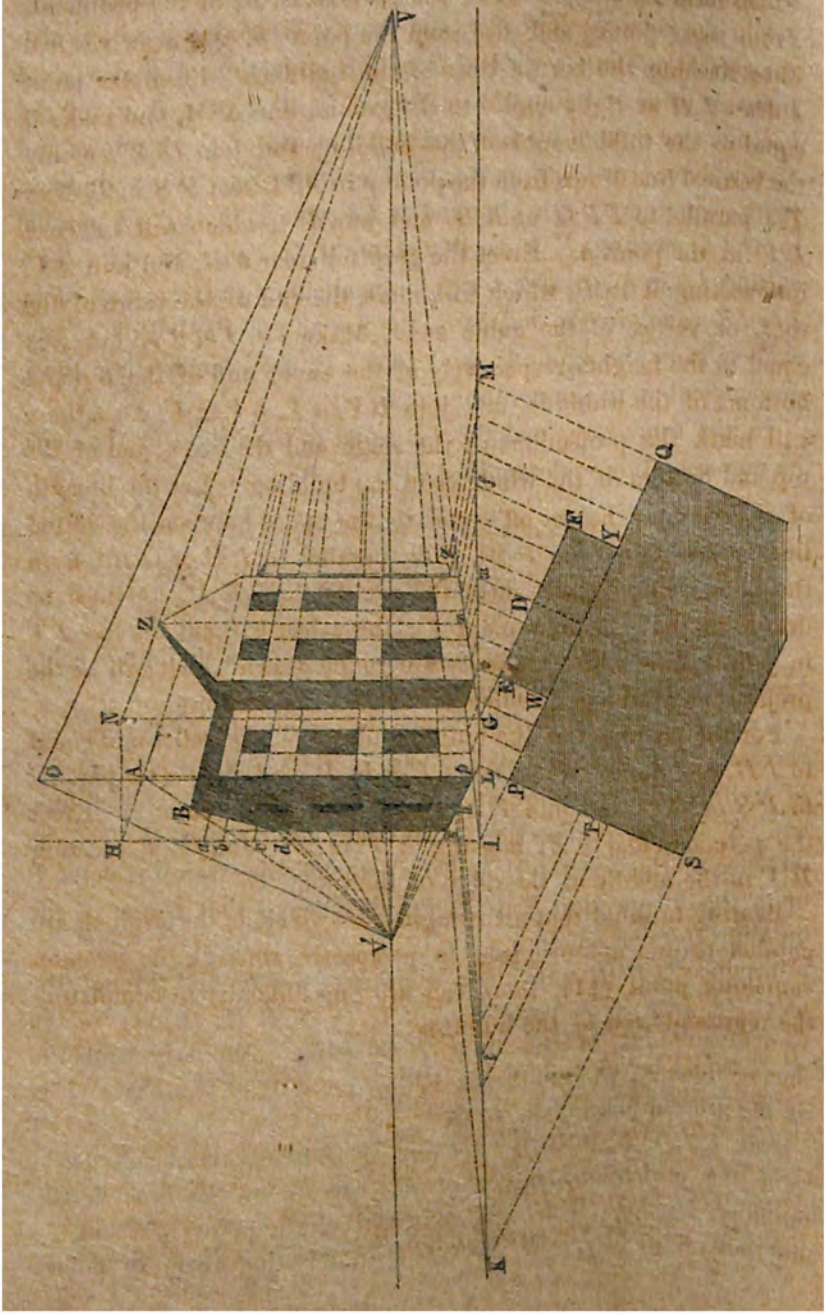
The dimensions of the different parts of a building being known, to draw the building in perspective.

Let $P\,W\,E\,F\,Y\,Q\,R\,S$, Fig. 33, be the ground plan of the building reversed, $E\ F$ the front of the base of the pediment, O the point of sight, C the centre of the picture, $V\ C\ V'$ the horizontal line, and $K\ M$ the intersection or ground line. Then $O\ C$ is the distance of the spectator, and the perpendicular from the point P upon the intersection $K\ M$ is the distance of that corner of the building from the picture.

Produce $Q\,P$, $R\,S$, to their intersections I, K, and through the point O draw $O\ V$ parallel to $I\ Q$ or $K\ S\ R$; then V is the vanishing point of the lines $P\ Q$, $S\ R$. Similarly find the intersections L, M, and the vanishing point V' of the other sides of the ground plan, $S\ P$, $R\ Q$.

Join $V\ I$, $V\ K$, $V'\ L$, $V'\ M$, and their intersections will determine the projections p, s, q, of the corners of the base of the building. Produce $W\,E$, $Y\,F$, and $F\,E$ to the points e, f, and G; and join $V'\ e$, $V'\ f$, and draw $G\ V$ intersecting them in points

Fig. 33.



which form the projections of the corners *E*, *F*, of the pediment. From these points, and also from the points *p*, *q*, *s*, draw vertical lines forming the corner lines of the building. From the point *I* draw *I H* at right angles to the ground line *K M*, and make it equal to the total height of the building, and join *H V* meeting the vertical line drawn from the point *p* in *A*. Bisect *P S* in *T*, draw *T t* parallel to *I P Q* or *K S*, and join *V t*, which will intersect *L V* in the point *h*. Erect the perpendicular *h B*, and join *A V* intersecting it in *B*, which will mark the end of the ridge of the roof, or vertex of the gable end. Make *I a*, *I b*, *I c*, *I d*, &c., equal to the heights respectively of the eaves and of the top and bottoms of the windows, and join *B V*, *a V*, *b V*, *c V*, &c.; these will mark the projections of the ridge and the eaves, and of the top and bottom of the windows in the building. Let the breadth of the windows be set off equal to the space between the dotted lines proceeding from *P W*, *Y Q*, parallel to *P S* or *Q R*; from the points where these meet the ground line *K M*, let lines be drawn to the vanishing point *V*; these last will cut the line *I V* in points from which verticals are to be drawn which will be the projections of the perpendicular sides of the windows.

For the pediment of the building, draw *G N* parallel and equal to *I H*, and join *N V*. Bisect *E F* in *D*, and draw *D x* parallel to *P S* or *R Q*, and join *V x*, cutting *G V* in the point *z*. Erect the perpendicular *z Z*, meeting *N V* in *Z*. Join *Z V* meeting *B V* in the middle of the ridge of the roof.

Bearing in mind that all straight lines which, in the original, are parallel to one another, pass in perspective through the common vanishing point (11), there can arise no difficulty in completing the representation of the building.

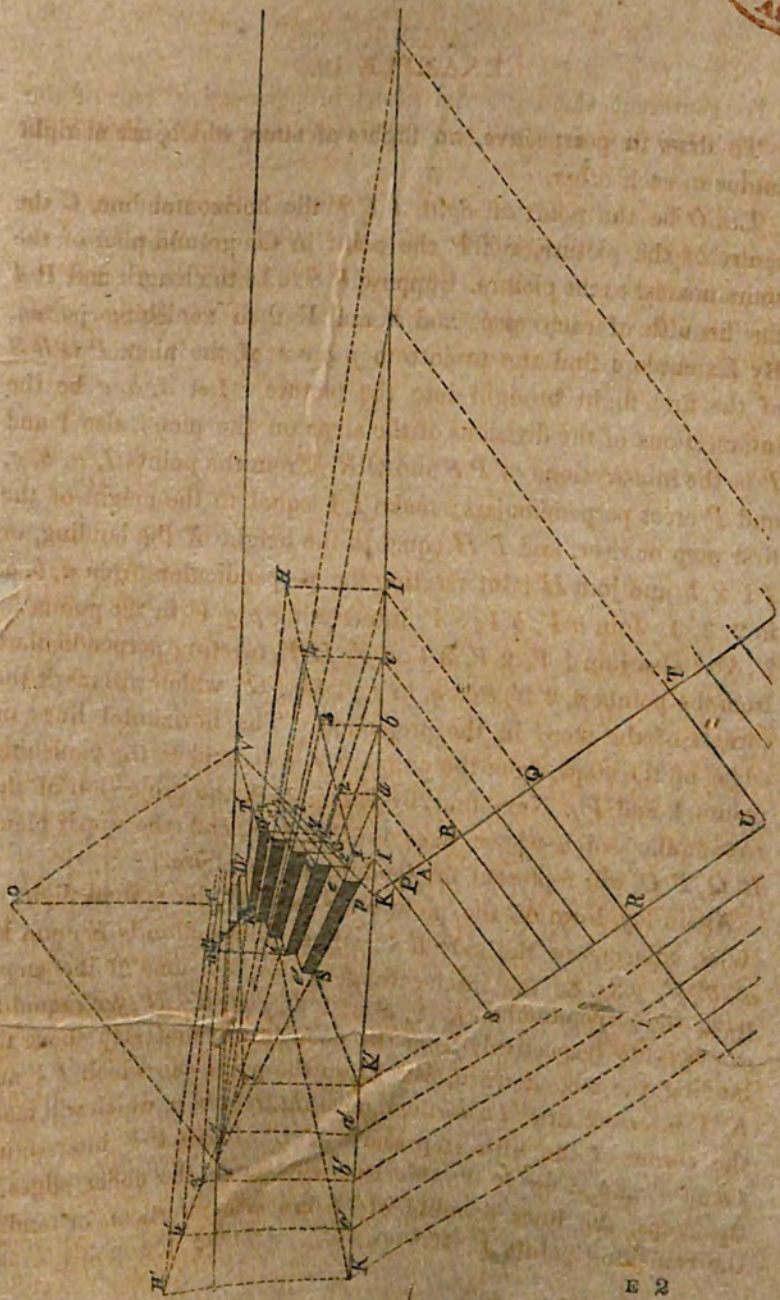
EXAMPLE 19.

To draw in perspective two flights of stairs which are at right angles to each other.

Let O be the point of sight, $V C V$ the horizontal line, C the centre of the picture, and P the point in the ground plan of the stairs nearest to the picture. Suppose $P S$ to be the length and $P A$ the breadth of each step, and V and V' their vanishing points. By Example 4 find the projection $p\ q\ r\ s$ of the plan $P Q R S$ of the first flight brought into the picture. Let a, b, c be the intersections of the divisions of the steps on the plan; also 1 and I' be the intersections of $P S$ and $Q R$. From the points I, a, b, c, and I' erect perpendiculars; make I 1 equal to the height of the first step or riser, and $I' H$ equal to the height of the landing, or I 1 × 5, and join H 1 intersecting the perpendiculars from a, b, c, in 2, 3, 4. Join $a V, b V, c V$, intersecting $p\ q\ V'$ in the points 2′, 3′, &c.; also join 1 V, 2 V, 3 V, 4 V, $H V$, meeting perpendiculars from the points p, 2′ 3′, &c., q, in e, f, &c., Q', which will mark the corners of the steps in the projection. The horizontal lines or edges of the steps, from the points e, f, &c., tend to the vanishing points V and V'. Next find, by Example 8, the projection of the rectangular solid, whose base is $R Q T U$, and the upper plane $R' Q' T' U'$ will represent the landing on the stairs.

Again lay down on the picture the plan of the second flight of stairs, adjacent to the side $R U$, whose intersection is K', and let a', b', c', &c., be the intersections of the divisions of the steps. Raise the perpendiculars K' 1, a' 2′, b' 3′, c' 4′, K'', H', &c., equal to the heights respectively from the floor of the first step above the landing, the second, third, &c. From the point i in which $I' V$ and $K' V'$ intersect, draw $i n$ vertical, meeting $H' V'$ in n, which will mark the corner of the fifth step above $R' U'$. Join 4′ V' intersecting $i n$ in u, and draw $R' u$; this line will touch the under edges of the steps, the lines forming which are either vertical, or tend to the vanishing points V, V'.

Fig. 34.



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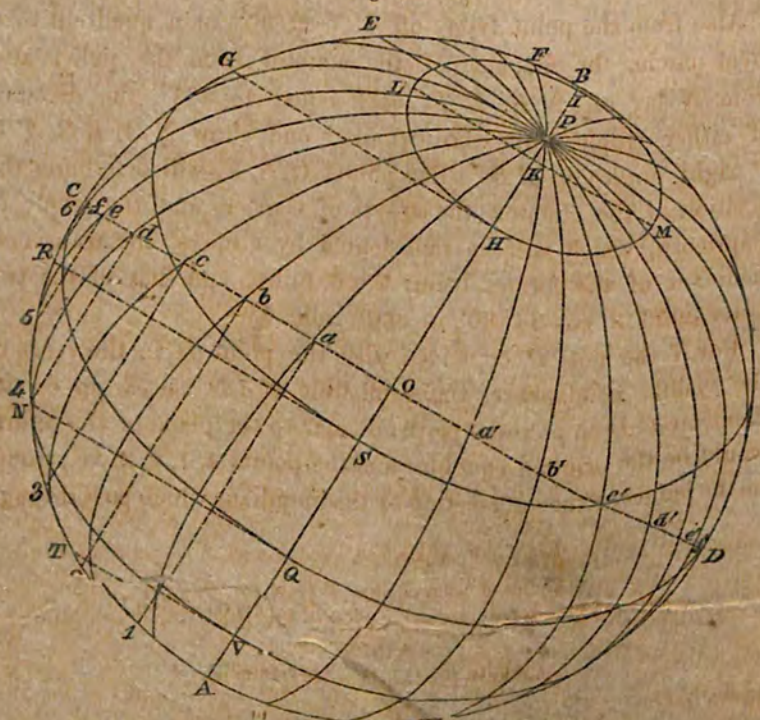
EXAMPLE 20.

To represent the terrestrial globe in perspective, one of the poles being just in sight, and the spectator at a considerable distance.

With the radius of the earth 3956 miles taken from any scale of equal parts, describe the circle $A\,D\,B\,C$, and draw the two diameters $A\,B$ and $C\,D$ at right angles to one another. Let the meridians be 24 in number, corresponding to the hour circles. Divide the quadrant $A\,C$ into $6\frac{1}{2}$ equal parts in the points

Fig. 35.



1, 2, 3, 4, 5, 6, and draw 1 a, 2 b, 3 c, 4 d, 5 e, 6 f perpendicular to $O\,C$. Take the distances $O\,a'$, $O\,b'$, $O\,c'$, &c., equal respectively to $O\,a$, $O\,b$, $O\,c$, &c. Assume the point P in

$A\,B$ to be the north pole visible, and through the points a, a', b, b', c, c', &c., and the pole P, describe ellipses whose common transverse axis shall be equal to $A\,B$ or $C\,D$, the given diameter of the earth, 7912 miles. Draw $P\,E$ at right angles $A\,B$, and lay off $E\,G$, $E\,F$, each equal to $23\frac{1}{2}°$, the distance of the arctic circle from the north pole, and draw $G\,H$, $F\,I$ perpendicular to $A\,B$. Bisect $I\,H$ in K, and make $K\,L$ and $K\,M$ perpendicular to $I\,H$, and each equal to the radius of the polar circle.

$$3956 \times \cos 66° 30' = 1577 \text{ miles.}$$

Through the four points I, L, H, M describe an ellipse, the rectangular axes being $L\,M$ and $I\,H$, and this will be the representation of the arctic or polar circle.

Also from the point E lay off $E\,N = 90°$ or a quadrant of a great circle, the distance of the equator from the pole; also from N lay off $N\,R$, $N\,T$, each equal to $23\frac{1}{2}°$, the distance of either tropic from the equator, and draw $N\,Q$, $R\,S$, $T\,V$ at right angles to $A\,B$. The points Q, S, V, will determine the position of the equator, the tropic of Cancer, and the tropic of Capricorn, which will be represented by ellipses, the transverse semiaxes of the former being 3956 miles, and that of the two latter $3956 \times \cos 23° 30'$, or 3628 miles*.

For if the pole P coincided with the point B in the verge of the visible hemisphere, the semicircle $C\,A\,D$ raised up on its diameter $C\,D$ so as to be perpendicular to the plane of the figure, would be the original equator; and the points A, 1, 2, 3, &c., would be the points of the intersection of the meridians which pass through

* Radius of polar circle = $3956 \times \cos 66° 30'$

Log rad. of polar circle = log 3956 + log cos 66° 30′ — 10

= 3·1979560.

∴ rad. of polar circle = 1577·43 miles.

Similarly,

Rad. of tropic of Cancer or of Capricorn = 3627·91 miles.

When the figure is required to be large, as for lectures or other purposes, we may avail ourselves without sensible error of the well-known method of a thread and two pins in describing the ellipses. The pins may be fixed in a thin flat ruler movable about the point O as a centre.

every 15° of longitude. These points projected upon the plane of the picture or figure will fall upon the points O, a, b, c, &c., as in the analogous case of a fluted column (Example 12). Now since the pole P is, by hypothesis, near the point B, those meridians or hour circles, passing through the pole will, without sensible error, pass also through the points O a, b, c, &c., a', b', c', &c., and be represented by ellipses (Example 7), whose common transverse axis is equal to $C D$, the diameter of the globe.

Again, if we take the line $A B$, passing through the centre of the picture O, to represent the meridian, whose plane passes through the eye and is perpendicular to the picture; and suppose the semicircle $A C B$ to be raised up on $A B$ so as to come into the position of that meridian, the point E will be the place of the original or real pole, which is visible, whose projection is the point P. Also the points F, G will be the originals of I, H, the projected points of the meridian intersected by the polar or arctic circle, which is represented by the ellipse $I L H M$. And since the arc $E N$ is a quadrant by construction, the projected point Q will represent the point of the equator intersected by that meridian. Similarly, as $N R$, $N T$ are each equal to $23\frac{1}{2}°$, the distance of the tropics from the equator, the points S, V, will represent the intersections of the projected meridian $A B$, with the tropic of Cancer and the tropic of Capricorn respectively.

The reader will probably consider the preceding examples sufficiently numerous and varied to make the practice of Perspective, in all ordinary cases, familiar to him.

PROBLEM IV.

To find the projection of any figure which is in a plane parallel to the picture.

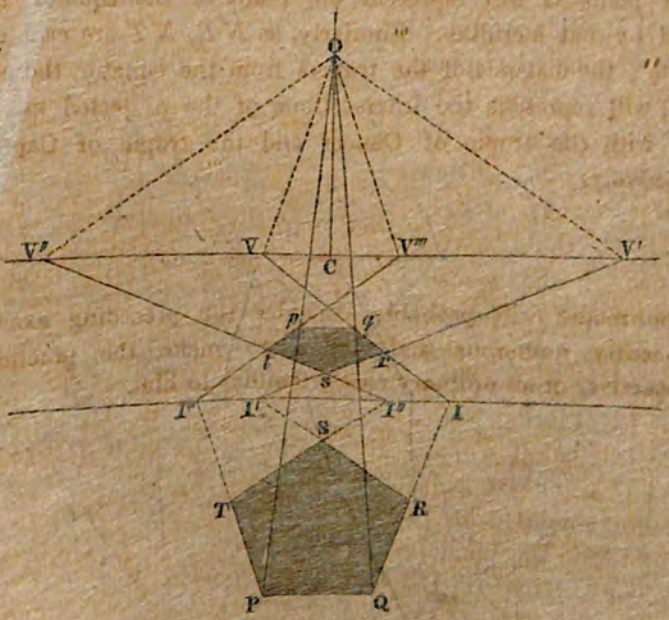
The projection being similar to its original (19) is obtained by making "an exact copy," that is, a figure similar to the original, the homologous sides being in the proportion of the distance of the eye from the plane of the picture to its distance from the plane of the original figure (20).

PROBLEM V.

Having given the intersection and vanishing line of a plane, with its centre and distance, to find the original of any given projection in the picture.

Let $V\,V'$ be the given vanishing line, C its centre, and $O\,C$ its

Fig. 36.



distance, and $I I'$ the given intersection of the plane; and suppose the figure $p q r s t$ to be the given projection, to find the original figure.

Produce the lines $q r$, $r s$, $s t$, $t p$, to meet the intersection in the points I, I', I'', I''', and the vanishing line in the points V, V', V'', V'''; join $O V$, $O V'$, $O V''$, $O V'''$, and parallel to these lines, through the points 1, 1′, 1″, 1‴ draw the lines $I R Q$, $I' S R$, $I'' S T$, $I''' T P$, intersecting in the points R, S, T. If the remaining side $p q$ be parallel to the vanishing line or intersection, join $O p$, $O q$, which will meet the lines $I''' T P$, $I R Q$ in the points P, Q; then $P Q$ being joined, the figure $P Q R S T$ will be the original required.

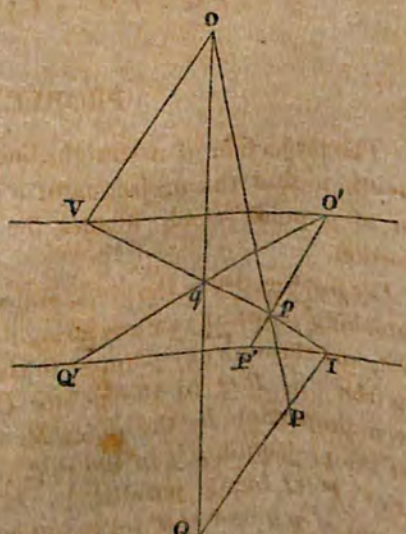
This construction is evident, being the reverse of Example 5. In the same manner we may go back to the original figure by using the directors as in Fig. 11.

PROBLEM VI.

The same things being given, to find the length only of the original of a given projection.

Let $p q$ be the given projection of a line, I the intersection, and V the vanishing point. Draw at pleasure any straight line $V O$, equal to the distance of the vanishing point, and $I Q$ parallel to it; join $O p$, $O q$, meeting $I Q$ in the points P, Q. $P Q$ is the length of the original of $p q$.

Fig. 37.



For O being the point of sight supposed to be brought into the plane of the picture along with the parallels $O V$ and $I Q$:

the straight lines OP, OQ, may be considered as visual rays from the original points P, Q, producing the projection pq.

Cor. By similar triangles IQq, VOq, and IPp, VOp, we have

$$Vq : VO :: Iq : IQ,$$
$$Vp : VO :: Ip : IP;$$

or,

$$IQ = \frac{Iq \cdot VO}{Vq}, IP = \frac{Ip \cdot VO}{Vp}$$

therefore,

$$PQ = \left(\frac{Iq}{Vq} - \frac{Ip}{Vp}\right) VO$$
$$= \frac{Iq \cdot Vp - Ip \cdot Vq}{Vq \cdot Vp} VO$$
$$= \frac{Ip \cdot Vp + pq \cdot Vp - Ip \cdot Vq}{Vq \cdot Vp} VO$$
$$= \frac{pq \cdot Ip + pq \cdot Vp}{Vq \cdot Vp} VO$$
$$= \frac{IV \cdot pq}{Vq \cdot Vp} VO.$$

Hence,

$$PQ : VO :: IV \cdot pq : Vq \cdot Vp.$$

PROBLEM VII.

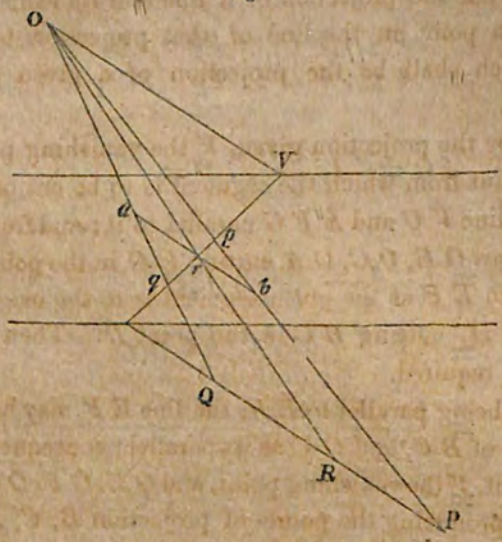
The projection of a straight line being given with its vanishing point, to find the projection of a given point in the original line, or of a point which divides the original line in a given proportion.

Let pq, Fig. 38, be the given projection of the line, and V the given vanishing point. Draw any straight line VO and draw PQ parallel to it, and through any point O in the line VO draw Op, Oq, meeting the line PQ in the points P and Q. Divide PQ in the given proportion in the point R, and join OR, which will intersect the projection pq in the required point r.

For PQ being parallel to OV, may be considered the original of pq and OV its parallel; consequently O is the point

of sight, V the vanishing point, and $O\,P, O\,R, O\,Q$, visual rays producing the points of projection p, r, q.

Fig. 38.



Cor. Through the point r draw $a\,b$ parallel to $O\,V$ or $P\,Q$, then by the similar triangles $q\,r\,a$, $q\,V\,O$, and $p\,r\,b$, $p\,V\,O$, we have

$$q\,r : q\,V :: a\,r : O\,V,$$
$$p\,V : p\,r :: O\,V : b\,r;$$

therefore,

$$q\,r . p\,V : q\,V . p\,r :: a\,r : b\,r$$
$$:: Q\,R : P\,R \text{ (by the parallel lines } a\,b, P\,Q);$$

or,

$$q\,r : p\,r :: q\,V . Q\,R : p\,V . P\,R.$$

Whence the point r may be found by dividing $p\,q$ in the point r in the given ratio $q\,V . Q\,R : p\,V . P\,R$.

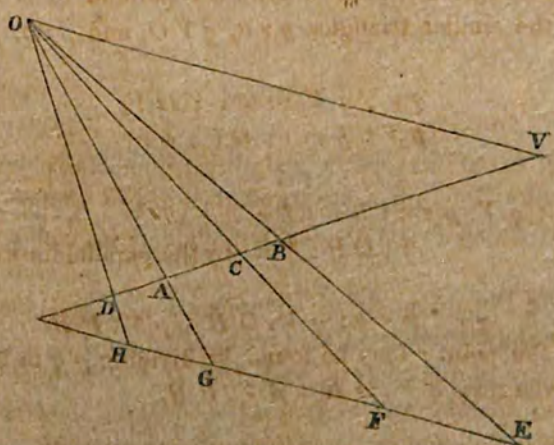
PROBLEM VIII.

Having given the projection of a line and its vanishing point, from a given point in the line of that projection to cut off a segment which shall be the projection of a given part of the original line.

Let $B\,C$ be the projection given, V the vanishing point, and A the given point from which the segment is to be cut off. Draw at pleasure the line $V\,O$ and $E\,F\,G$ parallel to it; and from any point O in $O\,V$ draw $O\,B$, $O\,C$, $O\,A$, cutting $E\,G$ in the points E, F, G. Make $G\,H$ to $E\,F$ as the given segment is to the original of $B\,C$, and draw $O\,H$, cutting $B\,C$ in the point D. Then will $A\,D$ be the segment required.

For $O\,V$ being parallel to $E\,F$, the line $E\,F$ may be considered the original of $B\,C$, and $O\,V$ as its parallel; consequently O is the point of sight, V the vanishing point, and $O\,E$, $O\,F$, $O\,G$, and $O\,H$, visual rays producing the points of projection B, C, A, and D.

Fig. 39.



When the point A coincides with either of the points B or C, this construction is reduced to the preceding, which may be considered as a particular case of this problem.

The demonstration, however, may be deduced from different

considerations. Instead of conceiving the point O to be the point of sight, OV may be considered as the vanishing line of a plane passing through the original of BC; and $EFGH$, being parallel to it, may be conceived to be the projection of some line in a plane, which is parallel to the picture (23); and therefore its parts EF and GH will be to one another in the same proportion as their originals (20). But since O is the common vanishing point, the originals of the projections OE, OF, OG, and OH are parallel to one another (16); therefore the original of AD is to the original of BC as GH to EF.

Cor. By the corollary to the last problem we have

$$AC.BV : BC.AV :: GF : EF,$$
$$AD.CV : AC.DV :: GH : GF;$$
$$\therefore AD.BV.CV : DV.BC.AV :: GH : EF;$$

or,

$$AD : DV :: GH.BC.AV : EF.BV.CV$$
$$:: m \qquad\qquad : n \text{ (suppose)}$$

Therefore the point D may be found by making AD to DV in the given ratio of m to n.

PROBLEM IX.

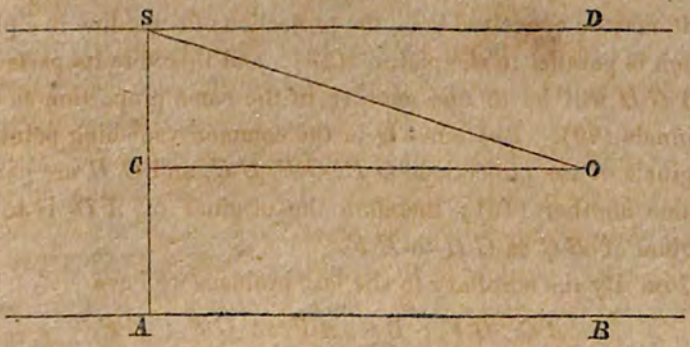
Having given the centre and distance of the picture, to find the vanishing line, its centre, and distance of a plane, the intersection of which is given, with the angle of its inclination to the picture.

Let AB, Fig. 40, be the given intersection of the plane, and C the centre of the picture. Draw CO parallel to AB, and equal to the distance of the picture, and draw CA perpendicular to AB. Draw OS, making the angle COS equal to the complement of the given inclination of the plane to the picture, and meeting AC in the point S, through which draw SD parallel to AB. Then SD will be the vanishing line sought, S its centre, and OS its distance.

If we suppose the triangle OCS to be raised up till its plane

is perpendicular to the picture, the point O will be the point of sight, and $S\,D$ being parallel to $A\,B$, a plane passing through the

Fig. 40.



line $S\,D$ and the point O will be parallel to the original plane, which passes through $A\,B$ and makes an angle with the picture, equal to $O\,S\,C$. Hence $S\,D$ is the vanishing line sought, and $O\,S$, being perpendicular to $S\,D$, is the distance of that vanishing line, and its centre the point S.

Let the inclined plane, whose intersection with the picture is the given line $A\,B$, make with the horizon an angle of 28°, or of 62° with the picture; and suppose $A\,C = 5$ feet, and the distance of the picture $O\,C = 12$ feet; then,

$$C\,S = O\,C \,.\, \cot O\,S\,C = 12 \,.\, \cot 62^\circ = 6{\cdot}38 \text{ feet.}$$
$$O\,S = O\,C \,.\, \operatorname{cosec} O\,S\,C = 12 \,.\, \operatorname{cosec} 62^\circ = 13{\cdot}59 \text{ feet.}$$
$$A\,S = 11{\cdot}38 \text{ feet.}$$

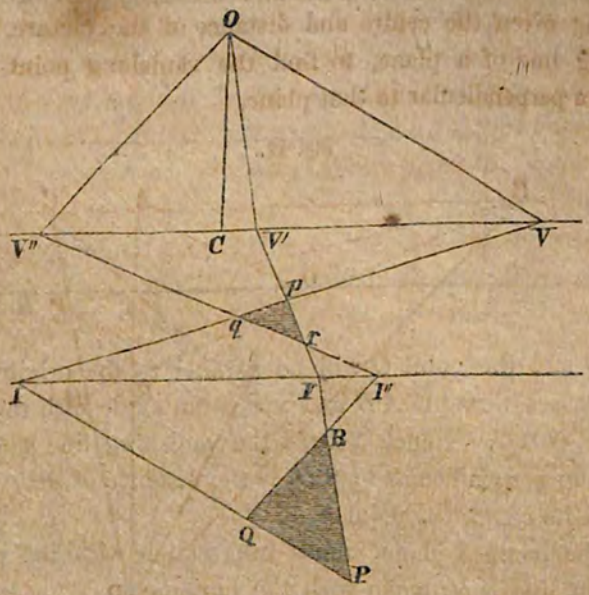
PROBLEM X.

Given the vanishing line of a plane, its centre and distance, and the projection of a line in that plane, to find the projection of another line in the plane, which makes a given angle with the former.

Let $C\,V$, Fig. 41, be the given vanishing line, C its centre, O the point of sight, consequently $O\,C$ is the given distance; also let $p\,q$ be the given projection: it is required to draw the line $p\,r$, so that

the original of the angle $q\,p\,r$ may be equal to the given angle. Produce $p\,q$ to its vanishing point V, join $O\,V$, and make the

Fig. 41.



angle $V\,O\,V'$ equal to the given angle; the point V' is the vanishing point through which the required projection must pass. Draw the straight line $V'\,p\,r$, which will be the line sought.

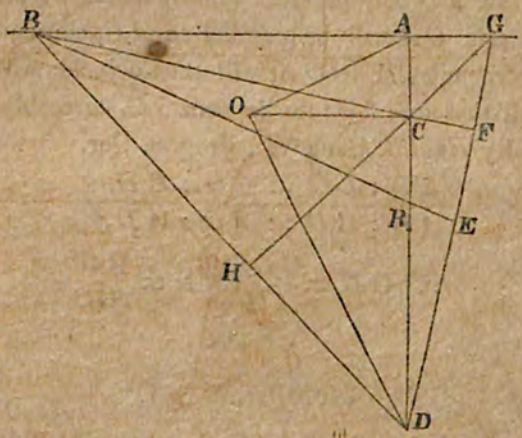
Let us suppose $P\,Q$, parallel to $O\,V$, to be the original of $p\,q$. Then $P\,R$, which is parallel to $O\,V'$, will be the original of $p\,r$. But $P\,Q$ and $P\,R$ being parallel to $O\,V$ and $O\,V'$, the angle $Q\,P\,R$ is equal to $V\,O\,V'$, the given angle. Wherefore (the angle $q\,p\,r$ representing the given angle $Q\,P\,R$) it is plain that the line $p\,r$ is properly determined.

If it had been required to make the angle $p\,q\,r$ to represent the angle $P\,Q\,R$, the angle $V\,O\,V''$ must have been made equal to the supplement of $P\,Q\,R$; for the angle $V\,O\,V'' = V\,O\,V' + V'\,O\,V'' = Q\,P\,R + P'\,R\,Q =$ the supplement of $P\,Q\,R$.

PROBLEM XI.

Having given the centre and distance of the picture, and the vanishing line of a plane, to find the vanishing point of lines which are perpendicular to that plane.

Fig. 42.



Let $A\,B$ be the vanishing line given, and C the centre of the picture. Draw $C\,A$ perpendicular to $A\,B$, and $C\,O$ parallel to it, and equal to the distance of the picture. Join $A\,O$ and draw $O\,D$ perpendicular to $A\,O$ cutting $C\,A$, produced in D; this will be the vanishing point sought.

Let the triangle $O\,A\,D$ be raised upon the plane of the figure, so that $O\,C$ may be at right angles to it, and the point O be brought into the point of sight. Then the plane passing through the point O and the line $A\,B$ will be parallel to the original plane (10), and the line $O\,D$ will be perpendicular to it, and consequently this line will be the parallel of lines perpendicular to the original plane; the point D will therefore be the vanishing point of those perpendicular lines, and $O\,D$ its distance.

Cor. 1. When the vanishing line $A\,B$ passes through the centre of the picture, that is, when the original plane is perpendicular to

the picture, the point D will be at an infinite distance, OD being drawn perpendicular to OA, now OC; and the projections of lines which are perpendicular to the original plane, will be perpendicular to AB, since they have to meet AC produced at an infinite distance in the line: consequently those projections will be parallel to one another; which indeed they ought to be, according to Art. 18, their originals being all parallel to the picture.

Cor. 2. When the original plane is parallel to the picture, the vanishing line AB will be at an infinite distance; therefore AC is infinite, and OA will be parallel to AC, consequently OD, which is perpendicular to OA, will then coincide with OC; the point D will therefore fall into the centre C, agreeably to Art. 16.

Cor. 3. By *Euclid*, Book VI., Prop. 8, Cor.

$$AC : CO :: CO : CD,$$
$$AC : AO :: AO : AD;$$
$$\therefore\ CD = \frac{CO^2}{AC},\ AD = \frac{AO^2}{AC}.$$

PROBLEM XII.

Given the centre and distance of the picture, and the vanishing point of certain parallel lines, to find the vanishing line of a plane which is perpendicular to those lines, also its centre and distance.

Let C, Fig 42, be the centre of the picture, and D the given vanishing point. Join CD, and draw CO perpendicular to it, and equal to the distance of the picture; also draw DO and OA perpendicular to it, meeting DC produced in A; then AB, drawn at right angles to AD, will be the vanishing line sought, A being its centre, and AO its distance.

For the triangle AOD being raised up on the line OD, so that CO may be at right angles to the plane of the figure, the line OD will be perpendicular to the plane passing through the point of sight O and the line AB, and consequently at right angles to its parallel planes. Therefore AB is the vanishing line of those planes (10), its centre being the point A, and AO its distance.

PROBLEM XIII.

The centre and distance of the picture being given, also the vanishing line of a plane, to draw through a given point the vanishing line of a plane which is at right angles to the former; and to find its centre and distance.

Let $A\,B$, Fig. 42, be the given vanishing line of the plane, and suppose C to be the centre of the picture, and E the given point. Find, by Problem XI., the vanishing point D of lines which are perpendicular to the plane of which $A\,B$ is the vanishing line. Join $D\,E$; this is the vanishing line sought.

Draw $C\,F$ perpendicular to $D\,E$, and make a right-angled triangle, whose base is $C\,F$, and the perpendicular from the point C equal to the given distance of the picture; then the hypothenuse will be the distance of the vanishing line $D\,E$, and the point F will be its centre.

For the plane of which the vanishing line is sought, is perpendicular to the plane the vanishing line of which is $A\,B$; therefore the vanishing line of that plane must pass through the vanishing point of all straight lines perpendicular to the plane of $A\,B$; viz. the point D, because some of these perpendiculars are in the first-mentioned plane. Now of the infinite number of those perpendicular planes whose vanishing lines pass through the point D, one of them must pass through the given point E; consequently $D\,E$ is the vanishing line sought.

And since $C\,F$ is drawn from the centre C at right angles to the vanishing line $D\,F$, the point F is its centre (13), and the distance of the point F from the point of sight placed at the distance of $C\,O$ drawn at right angles to the plane $A\,C\,F$, will be the distance of the vanishing line $D\,F$ (10).

Cor. 1. If $F\,C$ be produced to meet the given vanishing line $A\,B$ in B, this point will be the vanishing point of lines perpendicular to the plane of which $D\,E$ is the vanishing line. For that vanishing point is situated in the line $F\,C$, by the construction

of Problem XI., and it is in the given vanishing line AB, as appears from the preceding demonstration.

Cor. 2. And if the vanishing lines AB and DE meet in the point G, since it is well known that perpendiculars drawn from the three angles of a triangle to the opposite sides meet in one point, the points B, D, G will be the vanishing points of the three edges of the solid angle of a cube or of a right-angled parallelopiped. And consequently, if BD be joined, the lines BD, DG, GB will be the vanishing lines of the three planes which contain a solid right angle.

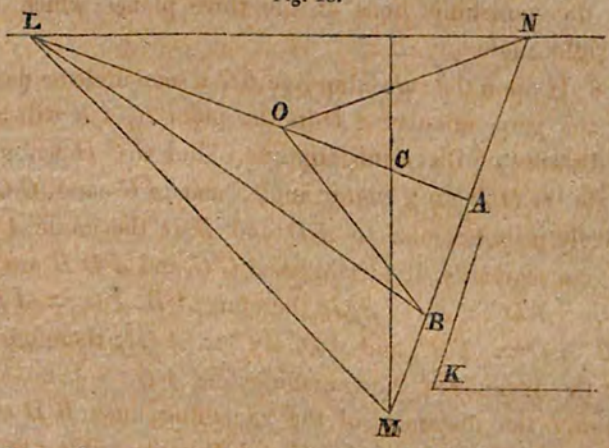
Cor. 3. If upon the vanishing line BG a semicircle be described, cutting the perpendicular AD in the point R, AR will be equal to the distance of that vanishing line. For GCH being drawn, the angle GHD is a right angle, and AG and CG being respectively perpendicular to AD and BD the angle $AGC = ADB$; consequently, the triangles AGC and ADB are similar, and $AB : AD :: AC : AG$; therefore $AB.AG = AD.AC$; but $AB.AG = AR^2$, and $AD.AC = AO^2$; therefore $AR = AO =$ the distance of the vanishing line AB.

Similarly the distances of the vanishing lines BD and DG may be set off from the points H and F by describing semicircles upon those lines.

PROBLEM XIV.

Having given the centre and distance of the picture, also the vanishing point of the common section of two planes inclined to each other in a given angle, with the vanishing line of one of the planes, to find the vanishing line of the other plane.

Fig. 43.



Let C be the centre of the picture, $L\,N$ the given vanishing line of one of the planes, L the vanishing point of their common section, and K the angle of inclination of the two planes. By Problem XII. find the vanishing line $M\,N$ of planes which are perpendicular to the lines whose vanishing point is L. In $L\,C\,A$, which is perpendicular to $M\,N$, take $A\,O$ equal to the distance of the vanishing line $M\,N$; join $O\,N$ and draw $O\,B$ making the angle $N\,O\,B$ equal to the given angle K; lastly, draw $B\,L$, which will be the vanishing line required. Imagine the triangle $O\,B\,N$ to be raised up on the line $N\,B$, so that the point O may come into the point of sight perpendicularly over the centre C of the picture, in which case the planes $L\,O\,N$, $N\,O\,M$, $M\,O\,L$ will be the parallels of the three original planes whose vanishing lines are $L\,N$, $N\,M$, $L\,M$. Now the plane whose vanishing line is $M\,N$ is perpendicular to the other two, for it is perpendicular to their

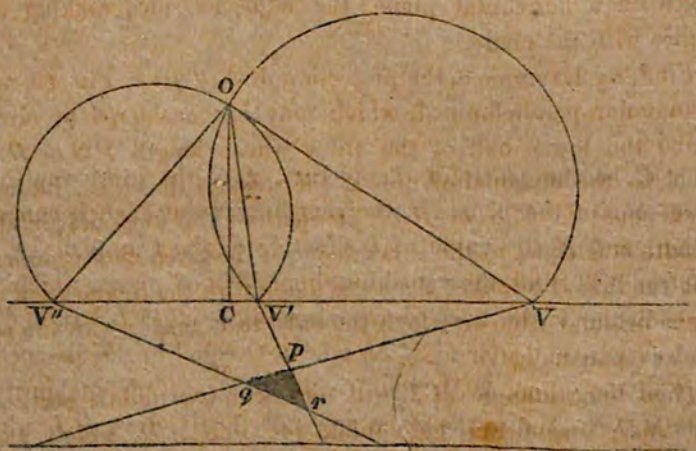
common section, whose vanishing point is L. Therefore the original planes whose vanishing lines are LN and LM are inclined to one another in the angle BON, that is, the angle K, since the angle of inclination of two planes is measured in a plane perpendicular to their common section. Hence, LN being the vanishing line given, BL is the vanishing line sought.

The centre of the vanishing line BL is found by drawing a line perpendicular to it from the point C, and its distance is found as in Problem XIII., Cor. 3.

PROBLEM XV.

The projection of a plane triangle of a known species being given, with its vanishing line, to find the centre and distance of that vanishing line.

Fig. 44.



Let pqr be the given projection, and VV' the vanishing line of the plane of the original triangle. Produce the sides of the projection to meet the vanishing line in the vanishing points V, V', V''. Upon VV' describe* the segment of a circle VOV' to contain an angle equal to that represented by qpr; and upon $V'V''$ a segment which shall contain the angle $V'OV''$ equal

* *Euclid*, Book III., Prop. 33.

to the original angle represented by $p\,r\,q$. From the point O, in which these circles cut one another, draw $O\,C$ at right angles to $V\,V'$; the point C will be the centre, and $O\,C$ the distance sought.

Supposing C to be the centre and $C\,O$ the distance of the vanishing line $V\,V'$, the originals of the angles $q\,p\,r$ and $p\,r\,q$ will be equal to $V\,O\,V'$ and $V'\,O\,V''$, by Problem X. Whence, these angles, by the construction being equal to the original angles represented by $q\,p\,r$ and $p\,r\,q$, the point C is the centre, and $C\,O$ is the distance of that vanishing line.

PROBLEM XVI.

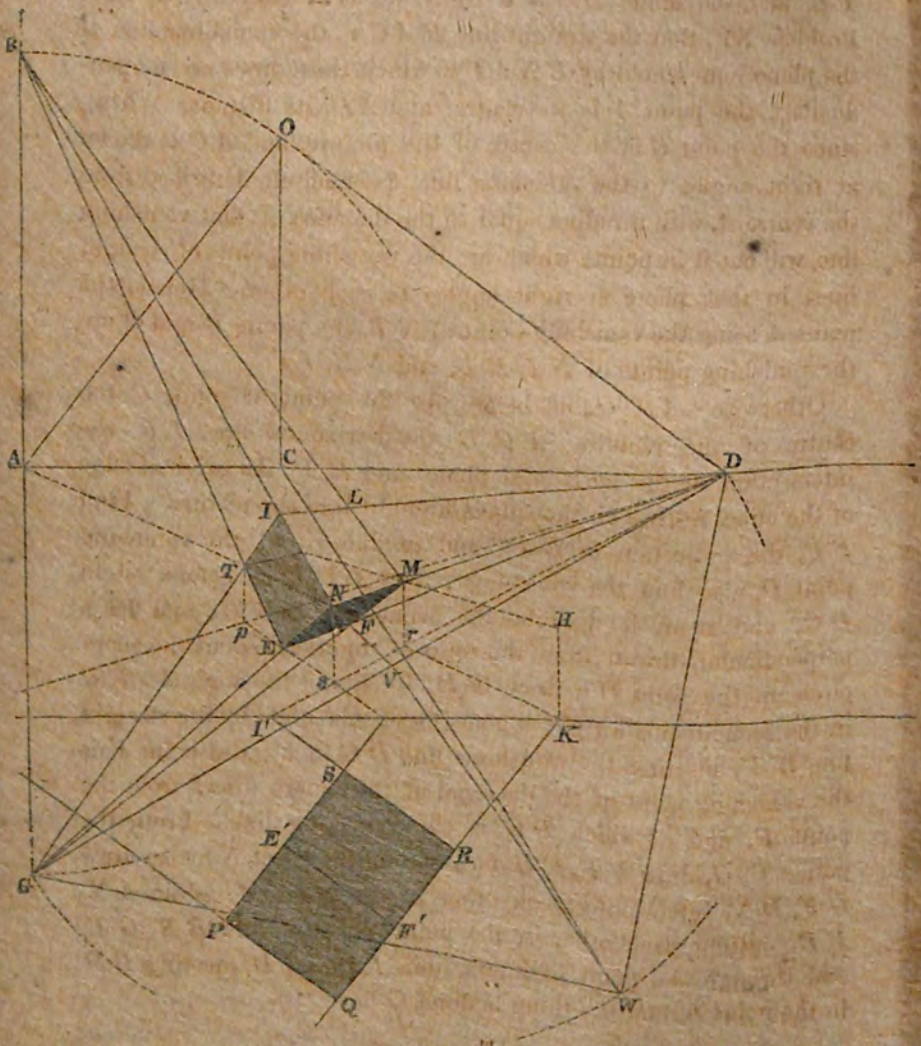
To draw a given cube in perspective, supported on one of its edges on a horizontal plane, the adjacent sides making equal angles with the plane.

Find, by Example 8, the projection $M\,N\,T\,p\,s\,r$, Fig. 45, of the rectangular parallelopiped, which may be considered to circumscribe the lower half of the cube, whose length $P\,S$ or $R\,Q$ is equal to the diagonal of one of the sides of the cube; its height $K\,H$ equal to $E''\,S$ or $R\,F''$, one-half of the diagonal; and the breadth $S\,R$ equal to the side or edge of the given cube; observing that the line $R\,S$ makes the same angle with the intersection $I\,K$ as the original edge on which the cube rests makes with the same line or its seat.

Then the rectangle $M\,T$ will represent the middle section of the cube; and being horizontal, the line $A\,D$, which passes through the centre C of the picture, is its vanishing line—the points A and D being the vanishing points of $N\,T$ and $N\,M$ and of their parallel sides in the section. Through the point A draw $B\,A\,G$ perpendicular to $A\,C\,D$, and O being the point of sight, from A as a centre, with the radius $A\,O$ describe a semicircle, cutting the line $B\,A\,G$ in the points B, G. Join $B\,N$, $B\,T$, $G\,N$, $G\,T$, intersecting in the points E and T respectively; join also $E\,D$, $G\,M$, intersecting each other in the point F; lastly, draw

B M and *D I* cutting each other in the point *L*. The figure *F E N M L I T* is the required representation of the given cube.

Fig. 45.



Since the line *T N* is the representation of the diagonal of one of the squares, and *N M* that of the side, also the distance of the

middle section of the cube from the edge on which it rests being equal to one-half of the diagonal, it is evident that the solid angles or points of the projection M, N, T, are correctly determined.

Again, since the point D is the vanishing point of $E\,F$, $N\,M$, $T\,L$, and the angle $D\,O\,A$ a right angle, it will be seen from Problem XI., that the straight line $B\,A\,G$ is the vanishing line of the plane represented by $E\,N\,I\,T$ to which those lines are perpendicular; the point A is its centre, and $A\,O$ its distance. Also, since the point C is the centre of the picture, and $A\,C$ is drawn at right angles to the vanishing line, a semicircle described from the centre A, with a radius equal to the distance of that vanishing line, will cut it in points which are the vanishing points of straight lines in that plane at right angles to each other. Hence, the point A being the vanishing point of $N\,T$, the points B and G are the vanishing points of $N\,I$, $E\,T$, and $N\,E$, $I\,T$.

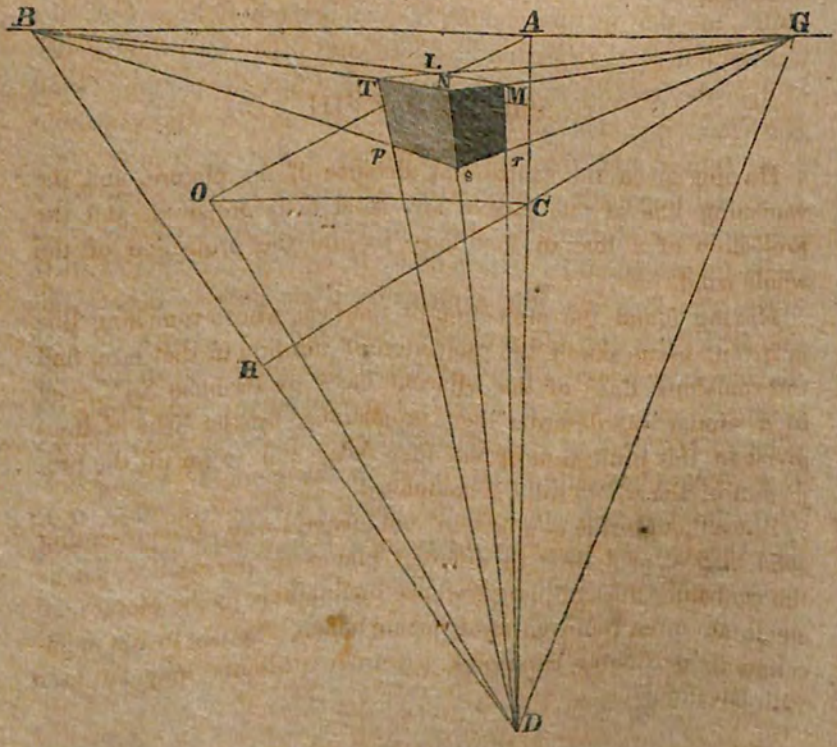
Otherwise,—Let O, as before, be the point of sight, C the centre of the picture, $A\,C\,D$ the horizontal line, $I\,K$ the intersection of the horizontal plane, and $E'\,F'$ the original edge of the cube, resting on that plane brought into the picture. Find $E\,F$, the projection of $E'\,F'$, and produce it to the vanishing point D, also find the vanishing points A, B, G as before. Join $D\,G$, and upon it describe the semicircle $D\,W\,G$, and let a perpendicular, drawn from the centre C upon $D\,G$, cut the semicircle in the point W; draw $W\,D$, $W\,G$, and the angle $D\,W\,G$ in the semicircle is a right angle. Bisect this angle by the straight line $W\,V$, meeting the vanishing line $D\,G$ in V; the point V is the vanishing point of the diagonal of the square drawn from the point F, and of which $D\,G$ is the vanishing line. From the points V, G, draw $V\,F$, $G\,E$ meeting in the point N; also draw $G\,F$, $D\,N$, intersecting each other in the point M. Join $A\,N$, $E\,B$, cutting each other in the point T; also draw $B\,N$, $G\,T$, and through the point of intersection I draw $I\,D$, meeting $B\,M$ in the point L, and the thing is done.

PROBLEM XVII.

Given the projection of a right-angled parallelopiped; to find its species and the centre and distance of the picture.

Let $L\,M\,N\,T\,p\,s\,r$ be the given projection. Produce the projections of the parallel sides to meet in their vanishing points G, B, and D; and join $G\,B$, $B\,D$, and $D\,G$, which will be the vanishing lines of the several faces of the solid whose species is sought (Problem XIII., Cor. 2). Draw $D\,A$ and $G\,H$ perpendicular to the vanishing lines $B\,G$ and $B\,D$ respectively, meeting in the point C. The point C is the centre of the picture. Upon $A\,D$ describe the semicircle $A\,O\,D$, and draw $C\,O$ perpendicular to $A\,D$, meeting the semicircle in O; the straight line

Fig. 46.



$C\,O$ will be the distance of the picture by Problem XI., since the angle $A\,O\,D$ in the semicircle is a right angle. Then find the distances of the vanishing lines $A\,B\,G\,D$ and $B\,D$ by Cor. 3, Problem XIII., and their centres A and H by Prop. II. Lastly, find the species of the originals of the faces $M\,T$, $T\,s$, and $M\,s$, by Problem V., considering them as divided into triangles by diagonals drawn in each face from the common point N.

Cor. When the vanishing line of one of the faces of the solid, $B\,G$, for instance, passes through the centre of the picture, the vanishing point D of the lines of the solid perpendicular to that face will be at an infinite distance; consequently, the edges $T\,p$, $N\,s$, and $M\,r$ will be parallel, as in Fig. 22, and the situation of $A\,D$ will be indeterminate, in which case the species of the face $L\,M\,N\,T$ may be taken at pleasure, and then the centre and distance of the picture may be found by retracing the steps of the usual construction.

PROBLEM XVIII.

Having given the centre and distance of the picture, and the vanishing line of one face of any solid body proposed, and the projection of a line in that face, to find the projection of the whole solid.

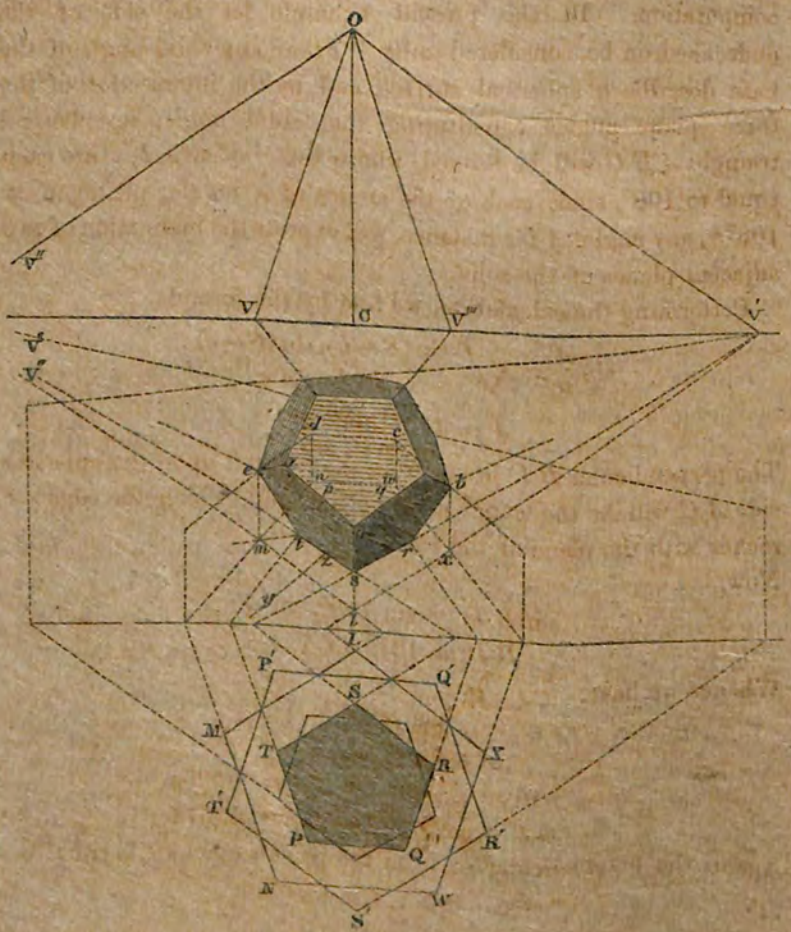
Having found the projection of the face, whose vanishing line is given, by means of the projection of the line in that face, find the vanishing lines of the adjacent faces by Problem XIV., and in a similar way describe their projections, by the help of lines given in the projection of the first face; and so on till the projection of the whole solid is completed.

But although this is a certain and general way of representing solid bodies, or figures in different planes in perspective, yet as the vanishing lines of planes, whose inclinations to the picture are small, are often removed to an inconvenient distance in the application to particular examples, several expedients may be used with advantage.

Example.—To determine the projection of a dodecahedron resting on one of its faces. This solid is one of the five regular bodies; it is contained by twelve equal and similar pentagons.

Let $P\,Q\,R\,S\,T$ be the base of the solid or the face of the dodecahedron on which it rests, O the point of sight, C the centre of the picture, and also of the vanishing or horizontal line $V\,C\,V'$. Let also the points $P', Q', R', S', T', L, M, N, W$, &c., be the seats or orthographic projections by perpendicular lines of the solid angles of the body upon the plane of the

Fig. 47.



base. The heights of these fifteen solid angular points must also be obtained either by measurement or otherwise. Find the projections of the base, and of the points $P', Q', R', S', T', L\,M\,N\,W$, &c., from which projections raise perpendiculars of the proper height, as determined by the vanishing points of the figure $P\,Q\,R\,S\,T$. Join the proper points, and the thing is done.

By considering and constructing certain parts separately, and erasing the subsidiary lines when done with, we may simplify the projection. But if the solid body is placed in such a position that we are unable to measure, we must avail ourselves of computation. In the present example let the side of the dodecahedron be considered unity. From any solid angle of the base describe a spherical surface, and by the intersection of the three plane angles constituting that solid angle, a spherical triangle $A\,B\,C$ will be formed, whose three sides a, b, c are each equal to 108°, since each of the angles of a regular pentagon = 108°*, any angle, A for instance, will express the inclination of two adjacent planes of the solid.

Performing the calculation, we find by the formula,

$$\sin\frac{A}{2} = \sqrt{\frac{\sin(S-b).\sin(S-c)}{\sin b\,.\,\sin c}},$$

$$A = 116°\,.\,33'\,.\,54''$$

The perpendicular $B\,D$ drawn from the point B upon the opposite side $A\,C$ will be the supplement of the angle which the edge $e\,t$ makes with the plane of the base.

Now,

$$\sin B\,D = \sin 108°.\sin A;$$
$$\therefore\ B\,D = 121°\,33'\,7'';$$

Whence we have,

$$e\,t\,m' = \pi - B\,D,$$
$$= 58°\,16'\,53'',$$
$$e\,m = \sin 58°\,16'\,53'' = \cdot 85064.$$
$$m\,t = \cos 58°\,16'\,53'' = \cdot 52574.$$

Again, the perpendicular $i\,z$ from the point i upon $s\,t$ is equal to

* *Euclid*, Book I., Prop. 32, Cor. 1.

the sum of the radii of the inscribed and circumscribed circles of the pentagon = 1·53884 (Mensuration),
whence,

$$1{\cdot}53884 \times \sin 63^\circ\, 26'\, 6'' = 1{\cdot}37635,$$

is the altitude of the point i above the plane of the base; and

$$1{\cdot}53884 \times \cos 63^\circ\, 26'\, 6'' = {\cdot}688222,$$

is the distance which the point i and the similar four points overhang the sides of the base of the solid.

Also the total height of the dodecahedron, or the distance between any two opposite parallel faces,

$$= m\,m' + 1{\cdot}37635$$
$$= 2{\cdot}22699.$$

If therefore the pentagon $P\,Q\,R\,S\,T$, whose side $P\,Q$, denoted by 1, is placed in the proper position in the picture to represent the side of the base of the given solid, place, concentric with it, an equal and similar pentagon in the position marked in the ground plan of the opposite or upper face of the solid. At the distance of ·688222 from the sides of these pentagons, draw lines parallel respectively to them, forming two other pentagons equal and similar, whose sides intersect each other in a manner like the former two.

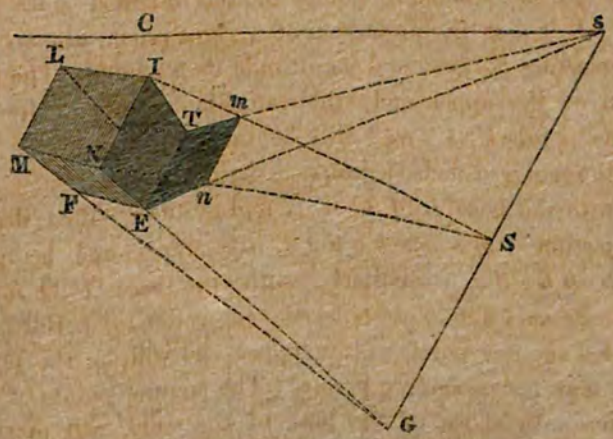
Find the projection $p\,q\,r\,s\,t$ of the base $P\,Q\,R\,S\,T$ by Example 5, the vanishing points of the inclined sides being $V, V', V'' V'''$; also the projection of a right prism, whose base is the intersecting polygon and its altitude = 2·22699. The representation of the upper end (the vanishing points of its sides being the points V, V', &c.) will be the projection of the upper face of the required dodecaedron.

Also with the base $L\,M\,N\,W\,X$ and altitude ·85064, describe the projection $l\,m\,n\,w\,x\,b\,c\,d\,e\,a$, the upper end being the pentagon $a\,b\,c\,d\,e$. Similarly, with the base $P'\,Q'\,R'\,S'\,T'$ and altitude = 1·37635 describe the projection of a third right prism, the angular points of the upper end will be the point i, and the four corresponding points. The angular points of these pentagons, with those of the base of the solid, will mark the position of the twenty solid angles of the dodecahedron.

The projection of a regular icosahedron supported on one of its points will be obtained by constructing in a similar manner two prisms upon equal and concentric pentagonal bases which intersect one another as in the original plane of Fig. 46.

It has not been our design to treat on the shadows of objects, although their figures are determined by the principles laid down in the preceding part of the work. When the light is supposed to come from the sun, the rays pass in parallel lines, and their common vanishing point must be obtained by observation. Let $FETILMN$ be the figure of a cube supported on its edge EF, CS be the vanishing line of the plane the cube rests on, and on which the shadow is cast; also let G be the vanishing point of the parallel edges NE, MF, IT, and S the given vanishing point of the sun's rays. Join GS and produce it to meet Cs in s; then the line GS, passing though the vanishing point G of the line EN, and also through the common vanishing point S of the sun's rays, is the vanishing line of the plane of those rays which pass through the line EN, or rather by whose interception by EN the shadow En is cast. Wherefore, since GS is the vanishing line of the plane of the rays passing through EN, and Cs is the vanishing line of the plane the shadow is cast on, it

Fig. 48.



follows by Prop. VII., Cor. 2, that the intersection s is the vanishing point of the common section of those planes, that is, of the shadow $E\,n$. Having, therefore, drawn $N\,S$, $E\,s$ cutting each other in the point n, the line $E\,n$ is the shadow of the edge $E\,N$. Draw also $I\,S$, $T\,s$, and their intersection m will determine $T\,m$ the shadow of $I\,T$. If $m\,n$ be joined, then the figure $E\,n\,m\,T$ will be the shadow of the plane $E\,T\,I\,N$. In the same way the shadow of the plane $L\,T$ is obtained; and similarly in other cases.

NOTE.

The figure at page 52 suggests the spherical form for children's humming-tops. If the five zones of the earth, with the meridians, were marked and painted on them, they would be extremely useful in education, and be equally amusing as toys. We may suppose the following dialogue to take place in a family where a lady is interested in teaching her children their first lesson in geography.

Mamma. See, my dears, what Papa has brought you, a new humming-top.

Children. O, how kind! Thank you, Papa. It is marked with lines like the large globes in the study, Papa.

Papa. Yes, and it hums and spins beautifully.

Mamma. Before you spin it you must examine it carefully, and then tell Papa what you learn from his kind present. Well, Albert, can you tell me of what form the earth is?

Albert. Round like an orange.

Alice. Or like this humming-top, Mamma.

Mamma. How large is the earth, or the globe on which we live?

Albert. Nearly 8000 miles through the middle.

Mamma. What is the axis of the earth?

Alice. The line about which the earth revolves, like a top spinning.

Mamma. What are the extremities of the earth's axis called?

Albert. The north and south poles of the earth.

Mamma. To what star in the heavens does the axis of the earth point?

Alice. The pole star.

Mamma. What is the equator?

Albert. The line which goes round the earth, equidistant from both poles.

Mamma. How far are the poles asunder?

Alice. 180 degrees.

Mamma. What distance, then, is the equator from each pole?

Albert. 90 degrees. A degree is 69 miles in length.

Mamma. Into how many parts or zones is the surface of the earth divided?

Alice. Into five zones:—the Torrid zone, two Temperate zones, and two Frigid zones.

Mamma. In what parts of the earth are the frigid zones situated?

Albert. They surround the poles, extending 23½° from each.

Mamma. What colour do you imagine the frigid zones to be? No answer.

Papa. The Frigid zone, derived from the Latin word *frigidus*, cold, must be white, being covered with ice and snow. It was the north frigid zone (the polar regions) which the brave Sir John Franklin and his crew left England to explore.

Mamma. Where is the North Temperate zone?

Alice. The zone or belt of the earth which joins the north frigid zone and extends to the Tropic of Cancer, or the parallel 23½° from the equator.

Mamma. What colour do you suppose the temperate zones to be? Look out at the window.

Children. The fields and trees are green.

Mamma. Yes; the climate in these parts being favourable to verdure and vegetation, the prevailing colour of the temperate zones will be green. Where is the Torrid zone, Albert?

Albert. The zone extending 23½° on each side of the equator.

Mamma. The native inhabitants of the torrid zone, from the excessive heat in those regions, are of a black or copper complexion.

Papa. The word torrid comes from *torridus*, hot, scorched.

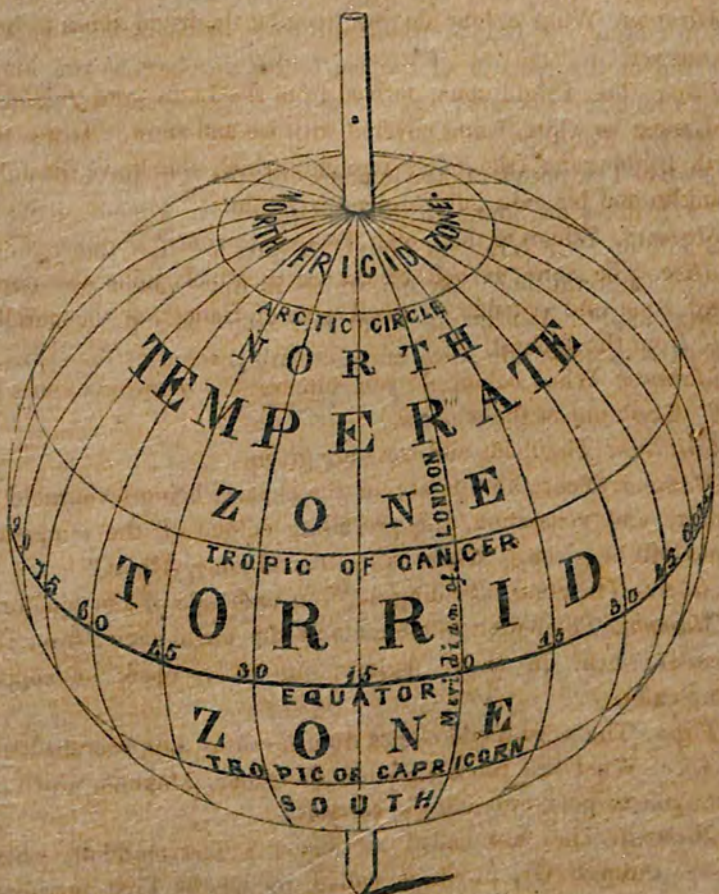
Alice. What are those lines on the globe, Mamma, which go from pole to pole, crossing the equator?

Mamma. They are called meridians. The meridian which passes through Greenwich is called by us the First meridian,

being the meridian from which the longitude is reckoned. The longitude of a place is the arc of the equator between the first meridian and the meridian of the place. Can you tell me, Albert, where the Arctic Circle is?

Albert. It is the circle which surrounds the north frigid zone, and is sometimes called the polar circle.

Papa. Quite correct. Well, you have answered very accurately; in the evening, when the lamp is on the table, I will explain to you, by means of the top and the lamp (which must be our earth and sun), the cause of the seasons, and the inequalities of days



and nights. Only one question more. What is the cause of the humming sound of tops when they are spinning?

Alice. I cannot tell. What is it, Papa?

Papa. When you spin the top the air within it acquires a rotatory motion, and by the centrifugal force it is thrown or blown, in the manner of a whistle, out at one side of the orifice. There is little doubt that various whistling notes might be produced by simple contrivances; but Mamma thinks their sound in the parlour is sufficiently loud. Suppose, Albert, a top is five inches in diameter, and in spinning it, 24 inches of string, in contact with the spindle, which is $\frac{1}{4}$ of an inch in diameter, comes equably off in a second of time:—which will move quicker, the equator of the top or the equator of the earth that revolves, as you know, once round in 24 hours?

Albert. The top.

Papa. The speed of the equator of the top is at the rate of $27\frac{1}{4}$ miles an hour: the inhabitants at the equator move at the rate of 1039 miles an hour. A question like this requires thought and a little calculation, which, in time, I hope you will be able to apply. You may now go and spin the top; Albert can hold the handle while Alice runs out with the end of the string.

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PRACTICE.

What will 378 lb. come to at $\frac{3}{4}d.$ per lb. ?

$\frac{1}{2}$	$\frac{1}{2}$	378*d.* price at 1*d.* per lb.
$\frac{1}{4}$	$\frac{1}{2}$	189*d.* price at $\frac{1}{2}d.$ per lb.
		$94\frac{1}{2}$ price at $\frac{1}{4}d.$ per lb.
12		$283\frac{1}{2}$ price at $\frac{3}{4}d.$ per lb.

23 $7\frac{1}{2}$. *Answer.*

What will 5624 yards cost at 1*s.* $9\frac{3}{4}d.$ per yard ?

6*d.*	$\frac{1}{2}$	5624*s.*	price at 1*s.* per yard.
3	$\frac{1}{2}$	2812	price at 6*d.* per yard.
$\frac{3}{4}$	$\frac{1}{4}$	1406	price at 3*d.* per yard.
		351 6	price at $\frac{3}{4}d.$ per yard.

20) 10193 6 price at 1*s.* $9\frac{3}{4}d.$ per yard.

£ 509 13 6. *Answer.*

Examples.

510 at $\frac{1}{4}d.$; 7314 at 1*s.* 8*d.*; 1072 at 1*s.* $5\frac{1}{2}d.$; 26 at 3*s.* 4*d.*; 6091 at $\frac{1}{2}d.$; 1322 at $11\frac{1}{2}d.$; 468 at 5*s.* $7\frac{3}{4}d.$

[*Continued on the back of the leaf.*

INTEREST,

COMMISSION, BROKERAGE, INSURANCE.

Interest is the sum charged for the loan of money. When the rate of interest is 3 per cent. per annum, it means that £3 is to be paid for the use of £100 for a year.

Commission or Brokerage is an allowance of so much per cent. for selling goods, &c., or transacting business for another. Insurance is a contract at so much per cent. made by one party to secure another party from risk.

Required the interest of £380 for 1 year at 2½ per cent. per annum.

What is the commission, or brokerage, or insurance, on £380 at 2½ per cent?

100 : 2½ : : 380

	2½
	760
	190
1,00	9,50
	20
	10,00

£9 10*s.* *Answer*

Examples.

Required the interest of £756 at 3 per cent. for 4 years.

[*Continued on the back*

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